SUMMATION OF SERIES

COLLECTED BY

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SECOND REVISED EDITION

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PREFACE TO DOVER EDITION

A SECOND edition published in the United States of America provides an opportunity for including many new series with an increase of more than 50 per cent over the original number. It has also been possible to rearrange the series in a more reasonable form.

Some corrections have been received from readers, and useful suggestions have been made by them for the present arrangement. These are gratefully acknowledged, and it will be of great assistance for future editions if readers will communicate their ideas for further expansion.

In using this collection himself, the author has experienced difficulty in tracing certain series, and there does not seem any solution excepting a complete search through all the series given. For example, certain series including inverse products appear in different parts of the book, and, if a search is to be avoided, a complete rearrangement combined with excessive duplication would be necessary. It does not seem possible, as in the case of a collection of integrals, to arrange them in a completely rational manner. Any suggestion in this direction would be specially welcome.

Among the new series included are some of those developed by Glaisher in many publications, notably the *Quarterly Journal of Mathematics*. Basing his series on Bernoulli functions, Glaisher evolved a number of coefficients which apparently simplify the appearance of the series. In this present collection, only a few are given, and the original articles should be consulted if the reader wishes to investigate them further.

The author wishes to acknowledge permission by the London Scientific Computing Service to publish tables from the *Index of Mathematical Tables* by Fletcher, Miller, and Rosenhead—an exceedingly useful book for any engaged in work on applied mathematics; and also to thank Mrs. H. M. Cooper for her excellent work in typing a difficult manuscript.

L. B. W. JOLLEY

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Richmond, Surrey, 1960

PREFACE TO FIRST EDITION

For a long time past there has been a need for a collection of series into one small volume for easy reference together with a bibliography indicating at least one of the textbooks to which reference could be made in case of doubt as to accuracy or to the method by which the series was arrived at.

The 700-odd series in this collection (with the exception of a few which have been specially prepared) are not new, and represent only the labour of extracting the material from the many textbooks on algebra, trigonometry, calculus and the like. Yet such a collection will, it is felt, be of considerable benefit to those engaged in the solution of technical problems, and will save a great deal of time in searching for the required result.

Criticism may be offered on the grounds that the inclusion of easy algebraical summations is unnecessary, but they have been inserted for a very definite purpose. For example, a series of inverse products may have for its sum an expression which is simple to find; but on the other hand, the solution may entail a complicated expression involving the integration or differentiation of other series. For this reason the arrangement of the series has been difficult, and overlapping is unavoidable in certain instances. To overcome this difficulty, the series have been set forth in as pictorial a manner as possible, so that the form of the individual terms can be readily seen.

On this account also, the inclusion of such series as are evolved for elliptic integrals, Bessel functions and the like has been restricted, perhaps to too great an extent; but reference to standard works is usually essential in such cases, and practically only such references are included.

The final column refers to the bibliography at the beginning of the book, and here again it has been quite impossible for obvious reasons to provide for all the references.

One of the most useful works, if it is desired to pursue any one particular problem further, is the *Smithsonian Tables.*

The scope of many of the series can be greatly enlarged by differentiation or integration of some of the forms given, and in the case of an integrated series, the constant of integration must be obtained by suitable methods. Infinite products are often of value in obtaining new series by taking logarithms and by differentiating or integrating subsequently.

In many cases it has been impossible in this small volume to comment on the limits or assumptions made in any particular summation; particularly is this the case with oscillating series: and in case of doubt it is always safer to refer to a textbook, and to bear in mind that this collection is supplementary to, and not in place of, the usual mathematical books.

Special attention is drawn in cases of difficult summations to the General and Special Forms (pages 216–225).

In all cases logh denotes the logarithm to the Napierian base, in accordance with modern practice.

Finally, any additions or corrections would be welcomed for embodiment in subsequent editions.

L. B. W. JOLLEY

Fairdene, Sheen Road,
Richmond, Surrey, 1925

CONTENTS

BIBLIOGRAPHY

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A†	T. J. Bromwich	*Introduction to the Theory of Infinite Series*, London: Macmillan Co., 1926.
B	L. L. Smail	*Elements of the Theory of Infinite Processes*, New York: McGraw-Hill Book Co., 1923.
C	G. Chrystal	*Algebra, An Elementary Text Book for the Higher Classes of Secondary Schools*, New York: Dover Publications, Inc., 1961.
D	Levett and Davison	*Plane Trigonometry*, New York: Macmillan Co., 1892.
E	S. L. Loney	*Plane Trigonometry* (Parts I and II), Cambridge: Cambridge University Press, 1900.
F	H. S. Hall and S. R. Knight	*Higher Algebra*, London: Macmillan Co., 1899.
G	E. T. Whittaker and G. Robinson	*Calculus of Observations*, Glasgow: Blackie and Son, 1937.
H	H. Lamb	*Infinitesimal Calculus*, Cambridge: Cambridge University Press, 1921.
J	L. Todhunter	*Integral Calculus*, London: Macmillan Co., 1880.
K	C. P. Steinmetz	*Engineering Mathematics*, New York: McGraw-Hill Book Co., 1911.
L	J. Edwards	*Differential Calculus for Beginners*, London: Macmillan Co., 1899.
M	J. Edwards	*Integral Calculus for Beginners*, London: Macmillan Co., 1898.
N	G. S. Carr	*Synopsis of Pure Mathematics*, London: Hodgson, 1886.

† In the text, the numbers preceding the *Reference* letter refer to the volume of the work cited; the numbers following the *Reference* letter refer to pages.

Indicating Letter	*Author*	*Title and Publisher*
O	E. W. Hobson	*A Treatise on Plane Trigonometry*, New York: Dover Publications, Inc., 1957.
P		*Encyclopaedia Britannica*, 11th edition.
Q	E. T. Whittaker and G. N. Watson	*Modern Analysis*, Cambridge: Cambridge University Press, 1920.
R	E. Goursat	*A Course in Mathematical Analysis*, Vol. 1, New York: Dover Publications, Inc., 1959.
T	E. P. Adams	*Smithsonian Mathematical Formulae*, Washington: Smithsonian Institute, 1922.
U	W. E. Byerly	*Fourier's Series*, New York: Dover Publications, Inc., 1959.
W	G. Boole	*Calculus of Finite Differences*, New York: Dover, 1960.
X	A. Eagle	*Fourier's Theorem*, New York: Longmans Green and Co., 1925.
Y	J. Edwards	*Differential Calculus*, London: Macmillan Co., 1938.
Z	J. Edwards	*Integral Calculus*, Vols. I and II, London: Macmillan Co., 1922.
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AB	H. S. Carslaw	*Fourier's Series and Integrals*, New York: Dover Publications, Inc., 1950.
AC	Fletcher, Miller, and Rosenhead	*Index of Mathematical Tables*, London: Scientific Computing Service, 1946.
AD	E. Jahnke and F. Emde	*Tables of Functions*, New York: Dover Publications, Inc., 1945.
AE	J. W. L. Glaisher	*Quarterly Journal of Mathematics*, Vol. 29, 1898.
AE.I.	J. W. L. Glaisher	*Quarterly Journal of Mathematics*, Vol. 28, 1896.
AG	E. W. Hobson	*The Theory of Functions of a Real Variable*, Vol. II, New York: Dover Publications, Inc., 1957.

SUMMATION OF SERIES

Series No.

I. Arithmetical Progression

(1) $a + (a + d) + (a + 2d) + \ldots n$ terms

II. Geometrical Progression

(2) $a + ar + ar^2 + \ldots n$ terms

(3) $a + ar + ar^2 + \ldots \infty$

(4) $1 + ax + a^2x^2 + a^3x^3 + \ldots \infty$

III. Arithmetical and Geometrical Progression

(5) $a + (a + d)r + (a + 2d)r^2 + \ldots n$ terms

(6) $a + (a + d)r + (a + 2d)r^2 + \ldots \infty$

(7) $1 + 2x + 3x^2 + 4x^3 + \ldots \infty$

(8) $1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \ldots n$ terms

(9) $1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \ldots n$ terms

(10) $1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \ldots \infty$

(11) $1 + 3x + 6x^2 + 10x^3 + \ldots \infty$

† See footnote to Bibliography.

Reference†

$$= \frac{n}{2}\{2a + (n - 1)d\} = \frac{n}{2}(a + l) \quad \text{where } l = \text{last term}$$ F. 29

$$= a\frac{(r^n - 1)}{r - 1}$$ F. 39

$$= \frac{a}{1 - r} \quad \text{where } r < 1$$ F. 40

$$= \frac{1}{1 - ax} \quad \text{where } ax < 1$$ F. 158

$$= \frac{a}{1 - r} + \frac{dr(1 - r^{n-1})}{(1 - r)^2} - \frac{\{a + (n - 1)d\}r^n}{1 - r}$$ F. 44

$$= \frac{a}{1 - r} + \frac{dr}{(1 - r)^2} \quad \text{where } r < 1$$ F. 44

$$= \frac{1}{(1 - x)^2} \quad \text{where } x < 1$$ F. 44

$$= \frac{35}{16} - \frac{12n + 7}{16 \cdot 5^{n-1}}$$ F. 45

$$= 4 - \frac{1}{2^{n-2}} - \frac{n}{2^{n-1}}$$ F. 45

$$= 6$$ F. 45

$$= \frac{1}{(1 - x)^3} \quad \text{where } x < 1$$ F. 45

Series No.

(12) $x(x + y) + x^2(x^2 + y^2) + x^3(x^3 + y^3) + \ldots n$ terms

(13) $\frac{2}{3} + \frac{3}{3^2} + \frac{2}{3^3} + \frac{3}{3^4} + \frac{2}{3^5} + \ldots \infty$

(14) $\frac{4}{7} - \frac{5}{7^2} + \frac{4}{7^3} - \frac{5}{7^4} + \ldots \infty$

(15) $1^2 + \frac{3^2}{2} + \frac{5^2}{2^2} + \frac{7^2}{2^3} + \ldots n$ terms

(16) $1 + 3x + 5x^2 + \ldots + (2n - 1)x^{n-1}$

IV. Powers of Natural Numbers

(17) $\sum_{x=1}^{x=n} x^p$

(18) $1 + 2 + 3 + 4 + \ldots n$

(19) $1^2 + 2^2 + 3^2 + 4^2 + \ldots n^2$

(20) $1^3 + 2^3 + 3^3 + 4^3 + \ldots n^3$

(21) $1^4 + 2^4 + 3^4 + 4^4 + \ldots n^4$

Reference

$$= \frac{x^2(x^{2n} - 1)}{x^2 - 1} + \frac{xy(x^n y^n - 1)}{xy - 1}$$ F. 46

$$= \frac{9}{8}$$ F. 46

$$= \frac{23}{48}$$ 0,4791$\dot{6}$ F. 46

$$= 34 - (4n^2 + 12n + 17)\frac{1}{2^{n-1}}$$

$$= \frac{1 + x - (2n + 1)x^n + (2n - 1)x^{n+1}}{(1 - x)^2}$$

$$= \frac{n^{p+1}}{p + 1} + \frac{n^p}{2} + \frac{1}{2}\binom{p}{1} B_1 n^{p-1} - \frac{1}{4}\binom{p}{3} B_2 n^{p-3}$$

$+ \frac{1}{6}\binom{p}{5} B_3 n^{n-5} - \ldots$ where $\binom{p}{n}$ are the binomial coefficients and B_n are Bernoulli numbers, see No. (1129). The series ends with the term in n if p is even, and with the term in n^2 if p is odd. T. 27

$$= \frac{n(n + 1)}{2}$$ F. 50

$$= \frac{n(n + 1)(2n + 1)}{6}$$ F. 50

$$= \left\{\frac{n(n + 1)}{2}\right\}^2$$ F. 51

$$= \frac{1}{30} n(n + 1)(2n + 1)(3n^2 + 3n - 1)$$ F. 256

Series No.

(22) $1^5 + 2^5 + 3^5 + 4^5 + \ldots n^5$

(23) $1^6 + 2^6 + 3^6 + 4^6 + \ldots n^6$

(24) $1^7 + 2^7 + 3^7 + 4^7 + \ldots n^7$

(25) $1^2 + 3^2 + 5^2 + 7^2 + \ldots n$ terms

(26) $1^3 + 3^3 + 5^3 + 7^3 + \ldots n$ terms

(27) $1^3 + (1.5)^3 + 2^3 + (2.5)^3 + \ldots \left(\frac{n+1}{2}\right)^3$

(28) $2^2 + 4^2 + 6^2 + 8^2 + \ldots n$ terms

(29) $1^2 \cdot 2^1 + 2^2 \cdot 2^2 + 3^2 \cdot 2^3 + \ldots n$ terms

(30) $1 \cdot 2^2 + 2 \cdot 3^2 + 3 \cdot 4^2 + \ldots n$ terms

(31) $(n^2 - 1^2) + 2(n^2 - 2^2) + 3(n^2 - 3^2) + \ldots n$ terms

(32) $\sum_1^n (2n - 1)^2$

(33) $\sum_1^n (2n - 1)^3$

(34) $\sum_1^n (x^n + n)(x^n - n)$

(35) $\sum_1^n (x^n + y^n)(x^n - y^n)$

(36) $\sum_1^n \left(x^n - \frac{1}{x^n}\right)\left(y^n - \frac{1}{y^n}\right)$

Reference

$$= \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^2}{12}$$ F. 337

$$= \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42}$$ F. 338

$$= \frac{n^8}{8} + \frac{n^7}{2} + \frac{7n^6}{12} - \frac{7n^4}{24} + \frac{n^2}{12}$$ F. 338

$$= \frac{1}{3} n(4n^2 - 1)$$ F. 256

$$= n^2(2n^2 - 1)$$ F. 256

$$= \frac{1}{8}\left\{\frac{(n+1)(n+2)}{2}\right\}^2 - \frac{1}{8}$$

$$= \frac{2n(n+1)(2n+1)}{3}$$

$$= 2^n\{2n^2 - 4n + 6\} - 6$$

$$= \frac{1}{12} n(n+1)(n+2)(3n+5)$$ F. 256

$$= \frac{1}{4} n^2(n^2 - 1)$$ F. 323

$$= \frac{1}{3} n(2n-1)(2n+1)$$

$$= n^2(2n^2 - 1)$$

$$= \frac{x^2(x^{2n} - 1)}{x^2 - 1} - \frac{1}{6} n(n+1)(2n+1)$$

$$= \frac{x^2(x^{2n} - 1)}{x^2 - 1} - \frac{y^2(y^{2n} - 1)}{y^2 - 1}$$

$$= \frac{\{(xy)^n - 1\}\{(xy)^{n+1} + 1\}}{(xy)^n(xy - 1)} - \frac{\left\{\left(\frac{x}{y}\right)^n - 1\right\}\left\{\left(\frac{x}{y}\right)^{n+1} + 1\right\}\frac{y}{n}}{\left(\frac{x}{y}\right)^n\left(1 - \frac{y}{x}\right)}$$

Series No.

(37) $\sum_{1}^{n} \frac{x^{n+1} - y^{n+1}}{x - y}$

(38) $\sum_{1}^{n} (a_0 n^r + a_1 n^{r-1} + \ldots a_r)$

See No. (17)

(39) $\sum_{1}^{\infty} x^n$

(40) $\sum_{1}^{\infty} nx^n$

(41) $\sum_{1}^{n} (1 + 2 + 3 \ldots + n)x^{n-1}$

V. Products of Natural Numbers

(42) To find the sum of n terms of a series, each term of which is composed of r factors in arithmetical progression, the first factors of the several terms being in the same arithmetical progression: Write down the nth term, affix the next factor at the end, divide by the number of factors thus increased, and by the common difference, and add a constant.

(43) $1 \cdot 3 \cdot 5 + 3 \cdot 5 \cdot 7 + 5 \cdot 7 \cdot 9 + \ldots n$ terms

(44) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 \ldots n$ terms

(45) $2 \cdot 5 + 5 \cdot 8 + 8 \cdot 11 + \ldots n$ terms

(46) $2 \cdot 2 + 4 \cdot 4 + 7 \cdot 8 + 11 \cdot 16 + 16 \cdot 32 \ldots n$ terms

(47) $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 \ldots n$ terms

(48) $1 \cdot 2 \cdot 3 \cdot 4 + 2 \cdot 3 \cdot 4 \cdot 5 + \ldots n$ terms

Reference

$$= \frac{x^2(1 - x^n)}{(x - y)(1 - x)} - \frac{y^2(1 - y^n)}{(x - y)(1 - y)}$$

$$= \{a_0 b_r + a_1 b_{r-1} + \ldots + a_r n\}$$
$$b_r = 1^r + 2^r + 3^r + \ldots n^r$$

$$= \frac{x}{1 - x} \quad \text{where } x < 1$$

$$= \frac{x}{(1 - x)^2} \quad \text{where } x < 1$$

$$= \frac{1 - x^n}{(1 - x)^3} - \frac{n(n + 3)x^n}{2(1 - x)^2} + \frac{n(n + 1)x^{n+1}}{2(1 - x)^2}$$

F. 314

$$= \frac{(2n - 1)(2n + 1)(2n + 3)(2n + 5)}{4 \cdot 2} + \frac{15}{8}$$

$$= n(2n^3 + 8n^2 + 7n - 2)$$ F. 315

$$= \frac{n(n + 1)(n + 2)}{3}$$ F. 52

$$= n(3n^2 + 6n + 1)$$ F. 318

$$= (n^2 - n + 4)2^n - 4$$ F. 333

$$= \frac{1}{4} n(n + 1)(n + 2)(n + 3)$$ F. 322

$$= \frac{1}{5} n(n + 1)(n + 2)(n + 3)(n + 4)$$ F. 322

Series No.

(49) $1\cdot4\cdot7 + 4\cdot7\cdot10 + 7\cdot10\cdot13 + \ldots$ n terms

(50) $1\cdot4\cdot7 + 2\cdot5\cdot8 + 3\cdot6\cdot9 + \ldots$ n terms

(51) $1\cdot5\cdot9 + 2\cdot6\cdot10 + 3\cdot7\cdot11 + \ldots$ n terms

(52) $6\cdot9 + 12\cdot21 + 20\cdot37 + 30\cdot57 + \ldots$ n terms
nth term is $(n + 1)(n + 2)(2n^2 + 6n + 1)$

(53) $2\cdot2 + 6\cdot4 + 12\cdot8 + 20\cdot16 + 30\cdot32 + \ldots$ n terms
nth term is $n(n + 1)2^n$

(54) $1\cdot3\cdot2^2 + 2\cdot4\cdot3^2 + 3\cdot5\cdot4^2 + \ldots$ n terms

(55) $\sum_1^n (p - n)(q - n)$

(56) $\sum_1^n \frac{m(m + 1)\ldots(m + n - 1)}{n!}$

(57) $\sum_1^n \frac{b(b + 1)(b + 2)\ldots(b + n - 1)}{a(a + 1)(a + 2)\ldots(a + n - 1)}$

(58) $\frac{1^2\cdot x}{2\cdot3} + \frac{2^2\cdot x^2}{3\cdot4} \ldots$ n terms

This series is integrable if $x = 4$.

(59) $\sum_1^n \frac{n^2 4^n}{(n + 1)(n + 2)}$

Reference

$$= \frac{1}{12}(3n - 2)(3n + 1)(3n + 4)(3n + 7) + \frac{56}{12}$$ F. 322

$$= \frac{1}{4}n(n + 1)(n + 6)(n + 7)$$ F. 322

$$= \frac{1}{4}n(n + 1)(n + 8)(n + 9)$$ F. 322

$$= \frac{2}{5}n(n + 1)(n + 2)(n + 3)(n + 4) + \frac{1}{3}(n + 1)(n + 2)(n + 3) - 2$$ F. 331

$$= (n^2 - n + 2)2^{n+1} - 4$$ F. 332

$$= \frac{1}{10}n(n + 1)(n + 2)(n + 3)(2n + 3)$$ F. 323

$$= \frac{n(n + 1)}{6}\{(2n + 1) - 3(p + q)\} + npq$$

$$= \frac{(m + 1)(m + 2)\ldots(m + 1 + n - 1)}{n!} - 1$$ C. 200

$$= \frac{(m + n)!}{m!\,n!} - 1$$

$$= \frac{b(b + 1)(b + 2)\ldots(b + n)}{(b + 1 - a)a(a + 1)(a + 2)\ldots(a + n - 1)} - \frac{b}{b + 1 - a}$$ T. 28

$$= \sum_{1}^{n} \frac{n^2x^n}{(n + 1)(n + 2)}$$ W. 58

$$= \frac{4^{n+1}}{3}\cdot\frac{n - 1}{n + 2} + \frac{2}{3}$$ W. 58

Series No.

VI. Figurate and Polygonal Numbers

(60) Figurate numbers—

1	1	1	1	1	1 . . .
1	2	3	4	5	6 . . .
1	3	6	10	15	21 . . .
1	4	10	20	35	56 . . .
1	5	15	35	70	126 . . .

The sum to n terms of the rth order

(61) Method of Differences⟹ due à Newton la pomme

One Series is	12	40	90	168	280	432 . . .
1st Diff.	28	50	78	112	152 . . .	
2nd Diff.	22	28	34	40 . . .		
3rd Diff.	6	6	6 . . .			
4th Diff.	0	0				

The nth term is The sum

$$12 + 28(n-1) + \frac{22(n-1)(n-2)}{2!} + \frac{6(n-1)(n-2)(n-3)}{3!}$$

(62) $4 + 14 + 30 + 52 + 80 + 114 + \ldots$ n terms

(63) $8 + 26 + 54 + 92 + 140 + 198 + \ldots$ n terms

(64) $9 + 16 + 29 + 54 + \ldots$ n terms

(65) $4 + 13 + 35 + 94 + 262 + \ldots$ n terms

(66) $2 + 12 + 36 + 80 + 150 + 252 + \ldots$ n terms

Reference

$$= \frac{1}{r!} n(n+1)(n+2)\ldots(n+r-1)$$ F. 320

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en général pour toutes les P.A.
si l'ordre de la p.a. est par example de 3
alors le nième terme est une équation du
(K-1) degré ; donc du second degré

$$= 12n + \frac{28n(n-1)}{2!} + \frac{22n(n-1)(n-2)}{3!} + \frac{6n(n-1)(n-2)(n-3)}{4!}$$

[redacted]

⟹ a vérifier.

$$= \frac{1}{12} n(n+1)(3n^2+23n+46)$$ F. 326

$$= n(n+1)^2$$ ⟹ pas bonne! F. 332

$$= \frac{1}{3} n(n+1)(5n+7)$$ F. 332

$$= 6(2^n-1) + \frac{1}{2} n(n+5)$$ F. 333

$$= \frac{3}{2}(3^n-1) + \frac{1}{6} n(n+1)(n+5) - n$$ F. 333

$$= \frac{1}{12} n(n+1)(n+2)(3n+1)$$ F. 332

Series No.

(67) $30 + 144 + 420 + 960 + 1890 + \dots\ n$ terms

(68) $2 + 5 + 13 + 35 + \dots\ n$ terms

(69) $2 + 7x + 25x^2 + 91x^3 + \dots\ n$ terms

VII. Inverse Natural Numbers

(70) $$\sum_1^n \frac{1}{n} = C + \operatorname{logh} n + \frac{1}{2n} - \frac{a_2}{n(n+1)} - \frac{a_3}{n(n+1)(n+2)} - \dots$$

Also $1 + \frac{1}{2} + \dots + \frac{1}{n}$

(71) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots\ \infty$

(72) $1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \dots\ \infty$

(73) $\left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right) + \dots \infty$

(74) $\frac{5}{1\cdot 2\cdot 3} + \frac{14}{4\cdot 5\cdot 6} + \dots\ \infty$

(75) $$2\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots\ \infty\right) = 8\left(\frac{1}{1\cdot 2\cdot 3} + \frac{1}{5\cdot 6\cdot 7} + \frac{1}{9\cdot 10\cdot 11} + \dots \infty\right)$$

Reference

$= \frac{1}{20} n(n + 1)(n + 2)(n + 3)(4n + 21)$ F. 332

$= \frac{1}{2} (3^n - 1) + 2^n - 1$ F. 272

$= \frac{1 - 4^n x^n}{1 - 4x} + \frac{1 - 3^n x^n}{1 - 3x}$ F. 272

where C = Euler's constant, see No. (1132),

$a_2 = \frac{1}{12} \quad a_3 = \frac{1}{12} \quad a_4 = \frac{19}{120} \quad a_5 = \frac{9}{20}$

and $a_k = \frac{1}{k} \int_0^1 x(1 - x)(2 - x) \ldots (k - 1 - x)\, dx$ T. 27

$= 7.48547$ where $n = 1000$ A. 325

$= 14.39273$ where $n = 10^6$

$=$ logh 2 = LOG 2 = LN 2 F. 195

$= 0.43882 = \frac{\pi}{4} - \frac{1}{2}$ logh 2 H. 475

$= \frac{1}{2}$ logh 2 C. 252

$= \sum_1^\infty \frac{9n - 4}{(3n - 2)(3n - 1)(3n)} =$ logh 3 C. 253

$=$ logh 4 C. 252

Logarithme hyperbolique
logarithme naturel
logarithme népérien
} same!

ε = e = 2,71828

Series No.

(76) $1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \ldots \infty$

(77) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots \infty$

$$= 1 - 2\left[\frac{1}{3\cdot 5} + \frac{1}{7\cdot 9} + \frac{1}{11\cdot 13} + \ldots \infty\right]$$

(78) $1 - \frac{1}{7} + \frac{1}{9} - \frac{1}{15} + \frac{1}{17} - \ldots \infty$

(79) $1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \ldots \infty$

(80) $\frac{1}{2} - \frac{1}{5} + \frac{1}{8} - \frac{1}{11} + \ldots \infty$

(81) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \ldots \infty$

(82) $1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \ldots \infty$

(83) $1 - \frac{1}{5} - \frac{1}{7} + \frac{1}{11} + \frac{1}{13} - \frac{1}{17} - \frac{1}{19} + \ldots \infty$

(84) $1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \ldots \infty$

(85) $1 - \frac{1}{2}\cdot\frac{1}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5} - \ldots \infty$

(86) $1 - \frac{1}{3} + \frac{1\cdot 2}{3\cdot 5} - \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7} + \ldots \infty$

(87) $\sum_{1}^{n} \frac{1}{2n-1}$

		Reference
$= \frac{\pi}{2\sqrt{2}}$	= 1,110720735	C. 335
$= \frac{\pi}{4}$	0,785398163	E. 109
$= \frac{\pi}{8}(1 + \sqrt{2})$	0,9480594490	M. 132
$= \frac{1}{3}\left(\frac{\pi}{\sqrt{3}} + \text{logh } 2\right)$	0,8356488483	A. 189
$= \frac{1}{3}\left(\frac{\pi}{\sqrt{3}} - \text{logh } 2\right)$	0,3735507277	A. 189
$= \frac{\pi}{3\sqrt{3}}$	0,6045997880	27 P. 281
$= \frac{1}{4\sqrt{2}}[\pi + 2\,\text{logh}(\sqrt{2} + 1)]$	0,8669729874	A. 190
$= \frac{1}{\sqrt{3}}\,\text{logh}(2 + \sqrt{3})$	0,7603459961	AB. 166
$= \frac{\pi}{2\sqrt{3}}$	= 0,9068996820	A. 528
$= \text{logh}(1 + \sqrt{2})$		Y. 90
$= \frac{2}{\sqrt{3}}\,\text{logh}\left(\frac{1 + \sqrt{3}}{\sqrt{2}}\right)$		Y. 90

$$= \frac{1}{2}(C + \text{logh } n) + \text{logh } 2 + \frac{B_1}{8n^2} - \frac{(2^3 - 1)B_2}{64n^4} + \ldots$$

For C see No. (1132) and for B_n see No. (1129) — A. 325

Series No.

(88) $\frac{x}{1+x} + \frac{2x^2}{1+x^2} + \frac{4x^4}{1+x^4} \dots \infty$

(89) $\frac{x}{1-x^2} + \frac{x^2}{1-x^4} + \frac{x^4}{1-x^8} + \dots$ *n* terms

(90) $\frac{x}{1-x^2} + \frac{x^2}{1-x^4} + \frac{x^4}{1-x^8} + \dots \infty$

(91) $\frac{1}{x+1} + \frac{2}{x^2+1} + \frac{4}{x^4+1} \dots \infty$

(92) $\frac{1-2x}{1-x+x^2} + \frac{2x-4x^3}{1-x^2+x^4} + \frac{4x^3-8x^7}{1-x^4+x^8} + \dots \infty$

(93) $\epsilon \left[\frac{1}{x} - \frac{1}{1!}\frac{1}{x+1} + \frac{1}{2!}\cdot\frac{1}{x+2} - \frac{1}{3!}\frac{1}{x+3} + \dots \infty\right]$

(94) $\frac{1}{3}S_2 - \frac{1}{5}S_4 + \frac{1}{7}S_6 - \dots \infty$

(95) $\frac{1}{2}\cdot\left(\frac{2x}{1+x^2}\right) + \frac{1}{2\cdot 4}\left(\frac{2x}{1+x^2}\right)^3 + \frac{1\cdot 3}{2\cdot 4\cdot 6}\left(\frac{2x}{1+x^2}\right)^5 + \dots \infty$

(96) $\sum_1^\infty 2x\left\{\frac{1}{n^2}\epsilon^A - \frac{1}{(n+1)^2}\epsilon^B\right\}$

$$A = -\frac{x^2}{n^2} \quad B = \frac{x^2}{(n+1)^2} \quad C = -x^2$$

VIII. Exponential and Logarithmic Series

(97) $1 + ax + \frac{a^2x^2}{2!} + \frac{a^3x^3}{3!} + \dots \infty$

(98) $1 + x \operatorname{logh} a + \frac{(x \operatorname{logh} a)^2}{2!} + \dots \infty$

(99) $x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \infty = x(1-x) + \frac{1}{2}x^2(1-x^2) + \dots \infty$

Reference

$= \frac{x}{1 - x}$ where $(x^2 < 1)$ T. 118

$= \frac{1}{1 - x} - \frac{1}{1 - x^n}$ A. 24

$= \frac{x}{1 - x}$ where $x^2 < 1$, and

$= -\frac{1}{x - 1}$ where $x > 1$ T. 118

$= \frac{1}{x - 1}$ where $x^2 > 1$ T. 118

$= \frac{1 + 2x}{1 + x + x^2}$ where $x < 1$ Y. 54

$= \frac{1}{x} + \frac{1}{x(x + 1)} + \frac{1}{x(x + 1)(x + 2)} + \ldots \infty$ A. 102

$= \frac{\pi \operatorname{logh} 2}{8}$ where $S_{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2n}$ A. 192

$= x$ where $|x| < 1$ Q. 132

$= 2x\epsilon^c$

AB. 167

$= \epsilon^{ax}$ F. 188

$= a^x$ F. 188

$= \operatorname{logh}(1 + x)$ where $x < 1$ F. 191

Series No.

(100) $-x-\frac{x^2}{2}-\frac{x^3}{3}-\ldots\infty$

(101) $1+2^3+\frac{3^3}{2!}+\frac{4^3}{3!}+\frac{5^3}{4!}+\ldots\infty$

(102) $\frac{1}{2}-\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}-\frac{1}{4\cdot 2^4}+\ldots\infty$

(103) $2\left\{\frac{1}{3!}+\frac{2}{5!}+\frac{3}{7!}+\ldots\infty\right\}$

(104) $\sum_{1}^{\infty}\frac{1}{n(4n^2-1)^2}$

(105) $(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\ldots\infty$

(106) $\frac{2}{1}+\frac{3}{2!}+\frac{6}{3!}+\frac{11}{4!}+\frac{18}{5!}+\ldots\infty$

(107) $\epsilon\left(1+x+\frac{2}{2!}x^2+\frac{5}{3!}x^3+\frac{15}{4!}x^4+\ldots\infty\right)$

(108) $1+2x+\frac{2^2-1}{2!}\cdot x^2+\frac{3^2-1}{3!}\cdot\frac{x^3}{2}+\ldots\infty$

(109) $1+\frac{2x^2}{2!}-\frac{x^3}{3!}+\frac{7x^4}{4!}-\frac{23x^5}{5!}+\frac{121x^6}{6!}-\ldots\infty$

(110) $2\left\{\frac{x^2}{2}-\frac{1}{3}\left(1+\frac{1}{2}\right)x^3+\frac{1}{4}\left(1+\frac{1}{2}+\frac{1}{3}\right)x^4-\ldots\infty\right\}$

(111) $2\left\{\frac{x^2}{2}+\frac{1}{3}\left(1+\frac{1}{2}\right)x^3+\frac{1}{4}\left(1+\frac{1}{2}+\frac{1}{3}\right)x^4+\ldots\infty\right\}$

(112) $x^2+\left(1-\frac{1}{2}+\frac{1}{3}\right)\frac{x^4}{2}+\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}\right)\frac{x^6}{3}+\ldots\infty$

(113) $x^2+\left(1-\frac{1}{3}+\frac{1}{5}\right)\frac{x^6}{3}+\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}\right)\frac{x^{10}}{5}+\ldots\infty$

	Reference
$= \operatorname{logh}(1 - x) \quad \text{where } x < 1$	F. 191
$= 15\epsilon$	F. 339
$= \operatorname{logh} 3 - \operatorname{logh} 2$	F. 195
$= \epsilon^{-1}$	F. 196
$= \frac{3}{2} - 2 \operatorname{logh} 2$	C. 253
$= \operatorname{logh} x \quad \text{where } 0 < x \leqslant 2$	T. 124
$= 3(\epsilon - 1)$	F. 338
$= \epsilon^{\epsilon^{x}}$	T. 126
$= (1 + x)\epsilon^{x}$	F. 338
$= \epsilon^{x} - \operatorname{logh}(1 + x)$	F. 338
$= \{\operatorname{logh}(1 + x)\}^2 \quad \text{where } x < 1$	F. 191
$= [\operatorname{logh}(1 - x)]^2$	A. 191
$= -\operatorname{logh}(1 + x) \cdot \operatorname{logh}(1 - x)$	A. 191
$= \frac{1}{2}(\tan^{-1} x) \operatorname{logh} \frac{1 + x}{1 - x}$	A. 191

Series No.

(114) $\epsilon^{-\pi} + \epsilon^{-9\pi} + \epsilon^{-25\pi} + \ldots \infty$

(115) $\sum a_n \theta^n$

(116) $1 - \frac{1}{2}\left(1 + \frac{1}{2}\right) + \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) - \ldots \infty$

(117) $\sum_1^\infty \frac{1}{nx^n}$

(118) $\sum_1^\infty \frac{x^n}{n}$

(119) $\sum_1^\infty \frac{x^n}{n!}$

(120) $\sum_1^\infty \frac{x^n}{n(n+1)}$

(121) $x + \left(1 + \frac{1}{2}\right)x^2 + \left(1 + \frac{1}{2} + \frac{1}{3}\right)x^3 + \ldots \infty$

(122) $\sum_1^\infty \frac{(n-1)x^n}{(n+2)n!}$

(123) $\sum_1^\infty \frac{(1^3 + 2^3 + 3^3 + \ldots n^3)}{n!} x^n$

(124) $\sum_1^\infty \frac{1}{x^2 + n^2}$

(125) $\sum_1^\infty (-1)^n \frac{1}{x^2 + n^2}$

Reference

$$= \frac{(2^{1/4} - 1)\Gamma(\frac{1}{4})}{2^{11/4}\pi^{3/4}}$$

$\Gamma(\frac{1}{4})$ $= 3.6256$ T. 144

$= \epsilon^A$ where $A = \epsilon^\theta$ and a_{n+1}

$$= \left(a_n + a_{n-1} + \ldots \frac{a_{n-2}}{2!} + \ldots \frac{a_0}{n!}\right)\frac{1}{n+1}$$ Y. 111

$= \frac{\pi^2}{12} - \frac{1}{2}(\text{logh } 2)^2$ 0,5822405267 A. 520

$= \text{logh}\,\frac{x}{x-1}$ H. 460

$= \text{logh}\,\frac{1}{1-x}$ where $x < 1$

$= \epsilon^x - 1$ trivial.

$= 1 - \left(\frac{1}{x} - 1\right)\text{logh}\,\frac{1}{1-x}$ F. 338

$= \frac{1}{1-x}\text{logh}\,\frac{1}{1-x}$ where $x < 1$

$= \frac{\{(x^2 - 3x + 3)\epsilon^x - \frac{1}{2}x^2 - 3\}}{x^2}$ C. 236

$= \epsilon^x\left(x + \frac{7}{2}x^2 + 2x^3 + \frac{1}{4}x^4\right)$ C. 235

$= \frac{\pi}{2x}\cdot\frac{\epsilon^{\pi x} + \epsilon^{-\pi x}}{\epsilon^{\pi x} - \epsilon^{-\pi x}} - \frac{1}{2x^2}$ T. 135

$= \frac{\pi}{x}\cdot\frac{1}{\epsilon^{\pi x} - \epsilon^{-\pi x}} - \frac{1}{2x^2}$ T. 135

Series No.

(126) $$\frac{1}{2n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots \frac{1}{m-1} + \frac{1}{2m}$$
$$- \frac{1}{12}\left(\frac{1}{n^2} - \frac{1}{m^2}\right) + \frac{1}{120}\left(\frac{1}{n^4} - \frac{1}{m^4}\right) - \frac{1}{252}\left(\frac{1}{n^6} - \frac{1}{m^6}\right)$$
$$+ \frac{1}{240}\left(\frac{1}{n^8} - \frac{1}{m^8}\right) + \dots \infty$$

(127) $$\frac{1}{2}\operatorname{logh} n + \operatorname{logh}(n+1) + \dots + \operatorname{logh}(m-1)$$
$$+ \frac{1}{2}\operatorname{logh} m - \frac{1}{12}\left(\frac{1}{m} - \frac{1}{n}\right) + \frac{1}{360}\left(\frac{1}{m^3} - \frac{1}{n^3}\right)$$
$$- \frac{1}{1260}\left(\frac{1}{m^5} - \frac{1}{n^5}\right) + \dots \infty$$

(128) $$\sum_{1}^{\infty}\left[n \operatorname{logh}\left(\frac{2n+1}{2n-1}\right) - 1\right]$$

(129) $$x - \frac{2}{3}x^3 + \frac{2 \cdot 4}{3 \cdot 5}x^5 + \dots \infty$$

(130) $$1 + (x\epsilon^{-x}) + \frac{3}{2!}(x\epsilon^{-x})^2 + \frac{4^2}{3!}(x\epsilon^{-x})^3 + \frac{5^3}{4!}(x\epsilon^{-x})^4 + \dots \infty$$
$$= 1 + \sum_{1}^{\infty}\frac{(n+1)^{n-1}}{n!}(x\epsilon^{-x})^n$$

(131) $$\frac{x^3}{1 \cdot 3} + \frac{x^5}{3 \cdot 5} + \frac{x^7}{5 \cdot 7} + \dots \infty \quad \text{(if convergent)}$$

(132)† $$1 - 2(2-1)B_1\frac{x^2}{2} + 2(2^3-1)B_2\frac{x^4}{4!}$$
$$- 2(2^5-1)B_3\frac{x^6}{6!} + \dots \infty$$

(133)† $$1 + 2ax + (2a)^2\left\{B_2\left(\frac{1}{2} + x\right) + \frac{1}{2}B_1\right\} + \frac{(2a)^3}{2!}B_2\left(\frac{1}{2} + x\right)$$
$$+ \frac{(2a)^4}{3!}\left\{B_4\left(\frac{1}{2} + x\right) - \frac{1}{4}B_2\right\} + \dots \infty$$

† For values of $B_n(x)$, see No. (1146).

Reference

$= \operatorname{logh} \dfrac{m}{n}$ where m and n integers X. 141

$= m \operatorname{logh} \dfrac{m}{\epsilon} - n \operatorname{logh} \dfrac{n}{\epsilon}$ where m and n integers X. 141

$= \dfrac{1}{2}(1 - \operatorname{logh} 2)$ A. 526

$= \dfrac{1}{\sqrt{1 + x^2}} \operatorname{logh} \{x + \sqrt{1 + x^2}\}$ where $|x| \leqslant 1$ A. 197

$= \epsilon^x$ Y. 456

$= \dfrac{x^2 - 1}{4} \operatorname{logh} \dfrac{1 + x}{1 - x} + \dfrac{x}{2}$ 1Z. 165

$= \dfrac{2x}{\epsilon^x - \epsilon^{-x}}$ AE. 12

$= \dfrac{2a\epsilon^{2ax}}{\epsilon^a - \epsilon^{-a}}$ AE. 14

Series No.

(134)† $\frac{1}{2} + xB_2\left(\frac{1}{2}\right) + \frac{x^3}{3!}B_4\left(\frac{1}{2}\right) + \frac{x^5}{5!}B_6\left(\frac{1}{2}\right) + \dots \infty$

(135)† $\frac{1}{3} + 3xB_2\left(\frac{1}{3}\right) + \frac{(3x)^2}{2!}B_3\left(\frac{1}{3}\right) + \frac{(3x)^3}{3!}B_4\left(\frac{1}{3}\right) + \dots \infty$

(136)† $\frac{1}{6} + 6xB_2\left(\frac{1}{6}\right) + \frac{(6x)^2}{2!}B_3\left(\frac{1}{6}\right) + \dots \infty$

(137) $x - \left(1 + \frac{1}{2}\right)x^2 + \left(1 + \frac{1}{2} + \frac{1}{3}\right)x^3$

$- \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)x^4 + \dots \infty$

(138) $x - \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} - \dots \infty$

(139) $\operatorname{logh} 2 + \frac{1\cdot 1}{2\cdot 2}x^2 - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 4}x^4 + \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6}x^6 - \dots \infty$

(140) $1 - \frac{x}{2} - \frac{x^2}{12} - \frac{x^3}{24} - \dots \infty = \int_0^1 (1 - x)^t \, dt$

(141) $x + \frac{x^3}{3} - \frac{2}{3}\cdot\frac{x^5}{5} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{x^7}{7} - \dots \infty$

(142) $x - \frac{2}{3}x^3 + \frac{2\cdot 4}{3\cdot 5}x^5 + \dots \infty$

(143) $\frac{x^2}{2} - \frac{2}{3}\cdot\frac{x^4}{4} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{x^6}{6} + \dots \infty$

(144) $- x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{3} - \frac{x^7}{7} + \frac{x^8}{8} + \dots \infty$

(145) $\frac{x^2}{2} + \frac{2x^3}{3} + \frac{3x^4}{4} + \dots \infty$

† For values of $B_n(x)$, see No. (1146).

Reference

$$= \frac{1}{\epsilon^{x/2} + 1}$$ AE. 30

$$= \frac{1}{1 + e^{x} + \epsilon^{2x}}$$ AE. 35

$$= \frac{1}{1 + \epsilon^{x} + \epsilon^{2x} + \epsilon^{3x} + \epsilon^{4x} + \epsilon^{5x}}$$ AE. 43

$$= \frac{\operatorname{logh}(1 + x)}{1 + x}$$ R. 425

$$= \operatorname{logh}(x + \sqrt{1 + x^2}) \quad \text{where } -1 \leqslant x \leqslant 1$$ L. 78

$$= \operatorname{logh}(1 + \sqrt{1 + x^2}) \quad \text{where } x^2 < 1$$ T. 123

$$= \frac{x}{\operatorname{logh} \dfrac{1}{1 - x}}$$ A. 190

$$= \sqrt{1 + x^2}\, \operatorname{logh}\{x + \sqrt{1 + x^2}\}$$ A. 191

$$= \frac{1}{\sqrt{1 + x^2}} \operatorname{logh}\{x + \sqrt{1 + x^2}\} \quad \text{where } x < 1$$ A. 197

$$= \frac{1}{2}\{\operatorname{logh}(x + \sqrt{1 + x^2})\}^2$$ L. 77

$$= \operatorname{logh}(1 - x + x^2)$$ L. 79

$$= \frac{x}{1 - x} + \operatorname{logh}(1 - x) \quad \text{where } x < 1$$ F. 197

Series No.

(146) $\log h\, 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} - \ldots \infty$

(147)† $\frac{x}{2} - B_1(2^2 - 1)\frac{x^2}{2!} - B_2(2^4 - 1)\frac{x^4}{4!}$

$$- B_3(2^6 - 1)\frac{x^6}{6!} - \ldots \infty$$

(148)† $2\left\{B_1(2^2 - 1)\frac{x}{2!} - B_2(2^4 - 1)\frac{x^3}{4!} + B_3(2^6 - 1)\frac{x^5}{6!} - \ldots \infty\right\}$

(149) $\frac{x}{1 \cdot 2} + \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} + \ldots \infty$

(150) $x + \frac{x^5}{5!} + \frac{x^9}{9!} + \ldots \infty$

(151)‡ $\log h\, 2 + \log h\, 3 + \ldots \log h\,(n - 1) + \frac{1}{2}\log h\, n$

(This series is not convergent.)

(152)‡ $1 - \frac{x}{2} + \frac{B_1{}^*}{2!}x^2 - \frac{B_3{}^*}{4!}x^4 + \frac{B_5{}^*}{6!}x^6 + \ldots \infty$

(153) $1 + \frac{x}{2} - \frac{1}{12}x^2 + \frac{1}{24}x^3 - \frac{19}{720}x^4 + \ldots \infty$

(154) $1 + \frac{x^2}{2 \cdot 3} - \frac{x^4}{4 \cdot 5} + \frac{x^6}{6 \cdot 7} + \ldots \infty = 1 + \sum_{1}^{\infty} \frac{x^{2n}(-1)^{n-1}}{2n(2n+1)}$

(155) $x^3 + x^4 + \frac{x^5}{2} + \ldots \infty$ et puis! quoi?

(156) $\frac{x^r}{r!} - {}_rP_1\frac{x^{r+1}}{(r+1)!} + {}_{r+1}P_2\frac{x^{r+2}}{(r+2)!} - \ldots \infty$

† For B_1, B_2, etc., see No. (1129).

‡ For values of B_1 and $B_1{}^*$, etc., see No. (1129).

Reference

$= \operatorname{logh}(1 + \epsilon^x)$ H. 498

$= \dfrac{x}{\epsilon^x + 1}$ N. 1543

$= \dfrac{\epsilon^x - 1}{\epsilon^x + 1}$ N. 1544

$= 1 + \dfrac{1 - x}{x} \operatorname{logh}(1 - x)$ F. 338

$= \dfrac{1}{4}(\epsilon^x - \epsilon^{-x} - j\epsilon^{jx} + j\epsilon^{-jx})$ where $j = \sqrt{-1}$ F. 338

$= \dfrac{1}{2} \operatorname{logh}(2\pi) + n \operatorname{logh} n - n$

$+ \dfrac{B_1}{1 \cdot 2 \cdot n} - \dfrac{B_2}{3 \cdot 4 \cdot n^3} + \cdots + \dfrac{(-1)^{r-1} B_r}{(2r - 1) \cdot 2r \cdot n^{2r-1}} \cdots$ 1P. 612

$= \dfrac{x}{\epsilon^x - 1}$ 2Z. 123

$= \dfrac{x}{\operatorname{logh}(1 + x)}$ W. 243

$= \dfrac{1}{x} \tan^{-1} x + \dfrac{1}{2} \operatorname{logh}(1 + x^2)$

$= \operatorname{logh}(1 + x^3 \epsilon^x)$ Y. 80

$= \dfrac{[\operatorname{logh}(1 + x)]^r}{r!}$ where ${}_rP_k$ is the sum of all products k at a time, of the first r natural numbers Y. 80

Series No.

(157) $\frac{x}{2} - \frac{x^2}{24} + \frac{x^4}{2880} - \dots \infty$

(158) $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} - \frac{4x^5}{5} + \frac{x^6}{6} + \dots \infty$

(159) Reversion of Series.

$$y = x - b_1x^2 - b_2x^3 - b_3x^4 - \dots \infty$$

can become

$$x = y + C_1y^2 + C_2y^3 + C_3y^4 + \dots \infty$$

if

C_1

C_2

C_3

C_4

C_5

C_6

C_7

See Van Orstrand (*Phil. Mag.* **19**: 366.1910) for coefficients up to C_{12}.

(160) $1 - \frac{1}{2(n+1)} - \frac{1}{2\cdot 3(n+1)^2} - \frac{1}{3\cdot 4(n+1)^3} - \dots \infty$

(161) $1 + \frac{2^3}{2!} + \frac{3^3}{3!} + \frac{4^3}{4!} + \dots \infty$

(162) $\frac{1}{n^2} + \frac{1}{2n^4} + \frac{1}{3n^6} + \dots \infty$

(163) $\frac{1}{n} + \frac{1}{2n^2} + \frac{1}{3n^3} + \dots \infty$

Reference

$= \operatorname{logh} \dfrac{x\epsilon^x}{\epsilon^x - 1}$ Y. 107

$= \operatorname{logh} (1 + x + x^2 + x^3 + x^4)$ Y. 107

T. 116

$= b_1$

$= b_2 + 2b_1^2$

$= b_3 + 5b_1b_2 + 5b_1^3$

$= b_4 + 6b_1b_3 + 3b_2^2 + 21b_1^2b_2 + 14b_1^4$

$= b_5 + 7(b_1b_4 + b_2b_3) + 28(b_1^2b_3 + b_1b_2^2) + 84b_1^3b_2 + 42b_1^5$

$= b_6 + 4(2b_1b_5 + 2b_2b_4 + b_3^2) + 12(3b_1^2b_4 + 6b_1b_2b_3 + b_2^3$
$+ 60(2b_1^3b_3 + 3b_1^2b_2^2) + 330b_1^4b_2 + 132b_1^6$

$= b_7 + 9(b_1b_6 + b_2b_5 + b_3b_4)$
$+ 45(b_1^2b_5 + b_1b_3^2 + b_2^2b_3 + 2b_1b_2b_4)$
$+ 165(b_1^3b_4 + b_1b_2^3 + 3b_1^2b_2b_3)$
$+ 495(b_1^4b_3 + 2b_1^3b_2^2) + 1287b_1^5b_2 + 429b_1^7$

$= \operatorname{logh} \left\{1 + \dfrac{1}{n}\right\}^n$ F. 197

$= 5\epsilon$ $\sum_{m \geq 0} \frac{m^k}{m!} = B_k \cdot e$ F. 197

$= 2 \operatorname{logh} n - \operatorname{logh} (n + 1) - \operatorname{logh} (n - 1)$ F. 197

$= \operatorname{logh} \dfrac{n}{n - 1}$ C. 368

Series No.

(164) $-\frac{x}{2}+\frac{5}{24}x^2-\frac{1}{8}x^3+\frac{251}{2880}x^4-\ldots\infty$

IX. Binomials. See also No. (1102).

(165) $x^n+nx^{n-1}a+\frac{n(n-1)}{2!}x^{n-2}a^2+\ldots+a^n$

(166) $1+\frac{1}{2}x^2+\frac{1\cdot 3}{2\cdot 4}x^4+\ldots\infty$

(167) $2+\frac{5}{3\cdot 2!}+\frac{5\cdot 7}{3^2\cdot 3!}+\frac{5\cdot 7\cdot 9}{3^3\cdot 4!}+\ldots\infty$

(168) $1-\frac{1}{2}\cdot\frac{1}{2}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{2^2}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{2^3}+\ldots\infty$

(169) $1+\frac{3}{4}+\frac{3\cdot 5}{4\cdot 8}+\frac{3\cdot 5\cdot 7}{4\cdot 8\cdot 12}+\ldots\infty$

(170) $1+\frac{1}{2}x-\frac{1\cdot 3}{2\cdot 4}x^2-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^4+\ldots\infty$

(171) $1-\frac{1}{2}x-\frac{1\cdot 3}{2\cdot 4}x^2+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^4-\ldots\infty$

(The above two series are useful in forming certain trigonometrical series.)

(172) $\frac{1}{3}x+\frac{1\cdot 4}{3\cdot 6}2^2x^2+\frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}3^2x^3+\ldots\infty$

(173) $1-x+x^2-x^3+\ldots\infty$

(174) $1-2x+3x^2-4x^3+\ldots\infty$

(175) $1+\frac{1}{2}x-\frac{1\cdot 1}{2\cdot 4}x^2+\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}x^3-\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4+\ldots\infty$

(176) $1-\frac{1}{2}x+\frac{1\cdot 3}{2\cdot 4}x^2-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^4-\ldots\infty$

Reference

$= \text{logh}\,\{\text{logh}\,(1 + x)^{1/x}\}$ Y. 107

$= (x + a)^n$

$= \dfrac{1}{\sqrt{1 - x^2}}$ H. 468

$= 3\sqrt{3}$ F. 167

$= \sqrt{2/3}$ F. 168

$= \sqrt{8}$ F. 168

$= \sqrt{\dfrac{\sqrt{1 + x^2} + x}{1 + x^2}}$

$= \sqrt{\dfrac{\sqrt{1 + x^2} - x}{1 + x^2}}$

$= \dfrac{x(x + 3)}{9(1 - x)^{7/3}}$

$= (1 + x)^{-1}$ T. 117

$= (1 + x)^{-2}$ T. 117

$= \sqrt{1 + x}$ T. 117

$= \dfrac{1}{\sqrt{1 + x}}$ T. 117

Series No.

(177) $1 + \frac{1}{3}x - \frac{1\cdot 2}{3\cdot 6}x^2 + \frac{1\cdot 2\cdot 5}{3\cdot 6\cdot 9}x^3 - \frac{1\cdot 2\cdot 5\cdot 8}{3\cdot 6\cdot 9\cdot 12}x^4 + \ldots \infty$

(178) $1 - \frac{1}{3}x + \frac{1\cdot 4}{3\cdot 6}x^2 - \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}x^3 + \frac{1\cdot 4\cdot 7\cdot 10}{3\cdot 6\cdot 9\cdot 12}x^4 - \ldots \infty$

(179) $1 + \frac{3}{2}x + \frac{3\cdot 1}{2\cdot 4}x^2 - \frac{3\cdot 1\cdot 1}{2\cdot 4\cdot 6}x^3 + \frac{3\cdot 1\cdot 1\cdot 3}{2\cdot 4\cdot 6\cdot 8}x^4 - \ldots \infty$

(180) $1 - \frac{3}{2}x + \frac{3\cdot 5}{2\cdot 4}x^2 - \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}x^3 + \ldots \infty$

(181) $1 + \frac{1}{4}x - \frac{3}{32}x^2 + \frac{7}{128}x^3 - \frac{77}{2048}x^4 + \ldots \infty$

(182) $1 - \frac{1}{4}x + \frac{5}{32}x^2 - \frac{15}{128}x^3 + \frac{195}{2048}x^4 - \ldots \infty$

(183) $1 + \frac{1}{5}x - \frac{2}{25}x^2 + \frac{6}{125}x^3 - \frac{21}{625}x^4 + \ldots \infty$

(184) $1 - \frac{1}{5}x + \frac{3}{25}x^2 - \frac{11}{125}x^3 + \frac{44}{625}x^4 - \ldots \infty$

(185) $1 + \frac{1}{6}x - \frac{5}{72}x^2 + \frac{55}{1296}x^3 - \frac{935}{31104}x^4 + \ldots \infty$

(186) $1 - \frac{1}{6}x + \frac{7}{72}x^2 - \frac{91}{1296}x^3 + \frac{1729}{31104}x^4 - \ldots \infty$

(187) $1 + n\left(\frac{x}{4}\right) + \frac{n(n-3)}{2!}\left(\frac{x}{4}\right)^2 + \frac{n(n-4)(n-5)}{3!}\left(\frac{x}{4}\right)^3 + \ldots \infty$

(188) $1 + \frac{n^2}{2!}x^2 + \frac{n^2(n^2-2^2)}{4!}x^4 + \frac{n^2(n^2-2^2)(n^2-4^2)}{6!}x^6 + \ldots$
$+ \frac{n}{1!}x + \frac{n(n^2-1^2)}{3!}x^3 + \frac{n(n^2-1^2)(n^2-3^2)}{5!}x^5 + \ldots \infty$

Reference

$= (1 + x)^{1/3}$ T. 117

$= (1 + x)^{-1/3}$ T. 117

$= (1 + x)^{3/2}$ T. 117

$= (1 + x)^{-3/2}$ T. 117

$= (1 + x)^{1/4}$ T. 117

$= (1 + x)^{-1/4}$ T. 117

$= (1 + x)^{1/5}$ T. 118

$= (1 + x)^{-1/5}$ T. 118

$= (1 + x)^{1/6}$ T. 118

$= (1 + x)^{-1/6}$ T. 118

$= \frac{1}{2^n}\{1 + \sqrt{1 + x}\}^n$ where $x^2 < 1$ and n is any real number T. 118

$= \{x + \sqrt{1 + x^2}\}^n$ where $x^2 < 1$ T. 118

Series No.

(189) $_mC_n = \dfrac{m(m-1)(m-2)\ldots(m-n+1)}{n!}$

(190) $1 + {}_mC_1x + \ldots {}_mC_nx^n + \ldots {}_mC_mx^m$

(191) ${}_{m_1}C_n + {}_{m_2}C_1 \cdot {}_{m_1}C_{n-1} + {}_{m_2}C_2 \cdot {}_{m_1}C_{n-2} + {}_{m_2}C_n$

(192) $1 + {}_mC_1 + {}_mC_2 + \ldots {}_mC_n + \ldots$

(193) $1 - {}_mC_1 + {}_mC_2 \ldots (-1)^n{}_mC_n + \ldots$

(194) $1 \cdot {}_mC_1 + 2 \cdot {}_mC_2x + \ldots n \cdot {}_mC_nx^{m-1} + \ldots$

(195) $m(m-1) + \dfrac{m(m-1)(m-2)}{1!} + \ldots$

$$+ \frac{m(m-1)\ldots(m-r+1)}{(r-2)!} + \ldots$$

(196) $1 + m + \dfrac{m(m+1)}{2!} + \ldots + \dfrac{m(m+1)\ldots(m+r-1)}{r!}$

(197) $2^n\left\{x^n + \dfrac{n}{1\cdot 2^2}x^{n-2}y^2 + \dfrac{n(n-3)}{2!2^4}x^{n-4}y^4 + \ldots\right.$

$$\left. + \frac{n(n-r-1)(n-r-2)\ldots(n-2r+1)}{r!2^{2r}}x^{n-2r}y^{2r} + \ldots\right\}$$

(198) $2^n(x^2 + y^2)^{1/2}$

$$\times \left\{x^{n-1} + \frac{n-2}{1!2^2}x^{n-3}y^2 + \frac{(n-3)(n-4)}{2!2^4}x^{n-5}y^4 + \ldots\right.$$

$$\left. + \frac{(n-r-1)(n-r-2)\ldots(n-2r)}{r!2^{2r}}x^{n-2r-1}y^{2r} + \ldots\right\}$$

(199) $1 - {}_mC_1 + {}_mC_2 - \ldots(-1)^n{}_mC_n$

(200) ${}_nC_1 - \dfrac{1}{2}{}_nC_2 + \dfrac{1}{3}{}_nC_3 - \ldots$

Reference

C. 186

$= (1 + x)^m$ C. 186

$= {}_{m_1+m_2}C_n$ C. 189

$= 2^m$ where $m > -1$ C. 191

$= 0$ where m is positive C. 191

$= m(1 + x)^{m-1}$ C. 197

$= m(m - 1)2^{m-2}$ where $m \not< 1$ C. 200

$= \dfrac{(m + r)!}{m!r!}$ where r is a positive integer C. 200

$= (x + \sqrt{x^2 + y^2})^n + \{x - \sqrt{x^2 + y^2}\}^n$ where n is a positive integer C. 204

$= \{x + \sqrt{x^2 + y^2}\}^n - \{x - \sqrt{x^2 + y^2}\}^n$ where n is a positive integer C. 205

$= (-1)^n{}_{m-1}C_n$ where n is a positive integer C. 210

$= 1 + \frac{1}{2} + \dots \frac{1}{n}$ C. 212

Series No.

X. Simple Inverse Products

(201)† To find the sum of n terms of a series, each term of which is composed of the reciprocal of the product of r factors in arithmetical progression, the first factors of the several terms being in the same arithmetical progression: Write down the nth term, strike off a factor from the beginning, divide by the number of factors so diminished, and by the common difference change the sign and add a constant.

(202) $\frac{1}{1.2.3.4} + \frac{1}{2.3.4.5} + \ldots n$ terms

(203) $\frac{3}{1.2.4} + \frac{4}{2.3.5} + \frac{5}{3.4.6} + \ldots n$ terms

(204) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots n$ terms

$+ \ldots \infty$

(205) $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots n$ terms

(206) $\frac{1}{2\cdot 3} + \frac{2}{3\cdot 4}\cdot 2 + \frac{3}{4\cdot 5}\cdot 2^2 + \ldots n$ terms

(207) $\frac{2}{1\cdot 3\cdot 4} + \frac{3}{2\cdot 4\cdot 5} + \ldots n$ terms

(208) $\frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + \frac{1}{7\cdot 10} + \ldots n$ terms

$$+ \ldots \infty = \sum_{1}^{\infty} \frac{1}{(3n-2)(3n+1)}$$

† In some cases the nth term can by partial fractions be resolved into the standard form when this rule can apply.

Reference

F. 316

$$= \frac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)}$$ F. 317

$$= \frac{29}{36} - \frac{1}{n+3} - \frac{3}{2(n+2)(n+3)} - \frac{4}{3(n+1)(n+2)(n+3)}$$ F. 317

$$= \frac{n}{n+1}$$ F. 322

$$= 1$$

$$= \frac{n}{2n+1}$$

$$= \frac{2^n}{n+2} - \frac{1}{2}$$

$$= \frac{17}{36} - \frac{6n^2 + 21n + 17}{6(n+1)(n+2)(n+3)}$$

$$= \frac{n}{3n+1}$$

$$= \frac{1}{3}$$ —ok [$\sum \phi(4/3, 8, 6)^{-1} = 1$] F. 322

Series No.

(209) $\frac{1}{1\cdot3\cdot5}+\frac{1}{3\cdot5\cdot7}+\frac{1}{5\cdot7\cdot9}+\ldots$ n terms

$+\ldots\infty$

(210) $\frac{1}{1\cdot4\cdot7}+\frac{1}{4\cdot7\cdot10}+\frac{1}{7\cdot10\cdot13}+\ldots$ n terms

$+\ldots\infty$

(211) $\frac{4}{1\cdot2\cdot3}+\frac{5}{2\cdot3\cdot4}+\frac{6}{3\cdot4\cdot5}+\ldots$ n terms

$+\ldots\infty$

(212) $\frac{1}{3\cdot4\cdot5}+\frac{2}{4\cdot5\cdot6}+\frac{3}{5\cdot6\cdot7}+\ldots$ n terms

$+\ldots\infty$

(213) $\frac{1}{1\cdot2\cdot3}+\frac{3}{2\cdot3\cdot4}+\frac{5}{3\cdot4\cdot5}+\ldots$ n terms

$+\ldots\infty$

(214) $\frac{3}{1\cdot2}\cdot\frac{1}{2}+\frac{4}{2\cdot3}\cdot\frac{1}{2^2}+\frac{5}{3\cdot4}\cdot\frac{1}{2^3}+\ldots$ n terms

(215) $\frac{1^2}{2\cdot3}\cdot4+\frac{2^2}{3\cdot4}\cdot4^2+\frac{3^2}{4\cdot5}\cdot4^3+\ldots$ n terms

(216) $\frac{1}{1\cdot3}+\frac{2}{1\cdot3\cdot5}+\frac{3}{1\cdot3\cdot5\cdot7}+\ldots$ n terms

(217) $\frac{1\cdot2}{3!}+\frac{2\cdot2^2}{4!}+\frac{3\cdot2^3}{5!}+\ldots$ n terms

(218) $\frac{1\cdot2}{3}+\frac{2\cdot3}{3^2}+\frac{3\cdot4}{3^3}+\ldots\infty$

(219) $\frac{5}{1\cdot2}\cdot\frac{1}{3}+\frac{7}{2\cdot3}\cdot\frac{1}{3^2}+\frac{9}{3\cdot4}\cdot\frac{1}{3^3}+\ldots$ n terms

Reference

$$= \frac{1}{12} - \frac{1}{4(2n+1)(2n+3)}$$

$$= \frac{1}{12}$$ F. 322

$$= \frac{1}{24} - \frac{1}{6(3n+1)(3n+4)}$$

$$= \frac{1}{24}$$ F. 322

$$= \frac{5}{4} - \frac{2n+5}{2(n+1)(n+2)}$$

$$= \frac{5}{4}$$ F. 322

$$= \frac{1}{6} - \frac{1}{n+3} + \frac{2}{(n+3)(n+4)}$$

$$= \frac{1}{6}$$ F. 322

$$= \frac{3}{4} - \frac{2}{n+2} + \frac{1}{2(n+1)(n+2)}$$

$$= \frac{3}{4}$$ F. 322

$$= 1 - \frac{1}{n+1}\cdot\frac{1}{2^n}$$ F. 333

$$= \frac{n-1}{n+2}\cdot\frac{4^{n+1}}{3} + \frac{2}{3}$$ F. 333

$$= \frac{1}{2} - \frac{1}{2}\,\frac{1}{1\cdot 3\cdot 5\cdot 7\ldots(2n+1)}$$ F. 333

$$= 1 - \frac{2^{n+1}}{(n+2)!}$$ F. 333

$$= \frac{9}{4}$$ F. 332

$$= 1 - \frac{1}{n+1}\cdot\frac{1}{3^n}$$ F. 331

Series No.

(220) $\frac{1}{3} + \frac{3}{3\cdot 7} + \frac{5}{3\cdot 7\cdot 11} + \dots n$ terms

(221) $\frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} + \dots \frac{1}{n(n+2)}$

(222) $\sum_{1}^{n} \frac{1}{(1+nx)(1+\overline{n+1}x)}$

(223) $\frac{1}{(x+1)(x+2)} + \frac{1!}{(x+1)(x+2)(x+3)} + \frac{2!}{(x+1)(x+2)(x+3)(x+4)} + \dots \infty$

(224) $\sum_{1}^{\infty} \frac{1}{(x+n)(x+n+1)}$

(225) $\frac{1}{a} + \frac{x}{a(a+1)} + \frac{x^2}{a(a+1)(a+2)} + \dots \infty$

(226) $\frac{a}{b} + \frac{a(a+1)}{b(b+1)}x + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}x^2 + \dots \infty$

(227) $\frac{1}{2n+2} + \frac{1}{2}\cdot\frac{1}{2n+4} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{2n+6} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{2n+8} + \dots \infty$

(228) $1 + \frac{a}{b} + \frac{a(a+1)}{b(b+1)} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)} + \dots \infty$

Reference

$$= \frac{1}{2} - \frac{1}{2}\frac{1}{3\cdot 7\cdot 11\ldots(4n-1)}$$ F. 331

$$= \frac{3n^2 + 5n}{4(n+1)(n+2)}$$ = 3/4 W. 57

$$= \frac{n}{(1+x)\{1 + \overline{n+1}x\}}$$

$$= \frac{1}{(1+x)^2}$$

élem → φ(M₂)^c 1/n

$$= \frac{1}{x+1}$$

φ(2,4,2)

$$= \frac{(a-1)!}{x^a}\left\{\epsilon^x - \sum_{n=0}^{a-1}\frac{x^n}{n!}\right\}$$ where a is positive T. 118

$$= (b-a)\binom{b-1}{a-1}\left\{\frac{(-1)^{b-a}\,\text{logh}\,(1-x)}{x^b}(1-x)^{b-a-1} + \frac{1}{x^a}\sum_{k=1}^{b-a}(-1)^k\binom{b-a-1}{k-1}\sum_{n=1}^{a+k-1}\frac{x^{n-k}}{n}\right\}$$ where a and b are positive and $a < b$; $\binom{b-1}{a-1}$, etc., are binomial coefficients T. 118

$$= \frac{2\cdot 4\cdot 6\ldots 2n}{3\cdot 5\cdot 7\ldots(2n+1)}$$ 1Z. 267

$$= \frac{b-1}{b-a-1}$$ where $b - 1 > a > 0$ A. 48

Series No.

(229) $\frac{a}{b} + \frac{2a(a+1)}{b(b+1)} + \frac{3a(a+1)(a+2)}{b(b+1)(b+2)} + \dots \infty$

(230) $\frac{1}{x(x+1)} + \frac{1}{x(x+1)(x+2)} + \frac{1 \cdot 2}{x(x+1)(x+2)(x+3)} + \dots \infty$

(231) $\frac{1}{(1+x)(1+2x)} + \frac{1}{(1+2x)(1+3x)} + \dots$ n terms

(232) $\frac{1}{(1+x)(1+ax)} + \frac{a}{(1+ax)(1+a^2x)} + \dots$ n terms

XI. Other Inverse Products

(233) $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \dots \infty$

(234) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 6} + \frac{1}{4\cdot 8} + \dots \infty$

(235) $\frac{1}{1\cdot 2} - \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} - \frac{1}{4\cdot 5} + \dots \infty$

(236) $\frac{1}{1\cdot 2\cdot 3} + \frac{1}{3\cdot 4\cdot 5} + \frac{1}{5\cdot 6\cdot 7} + \dots \infty$

(237) $\frac{1}{1\cdot 2\cdot 3} - \frac{1}{3\cdot 4\cdot 5} + \frac{1}{5\cdot 6\cdot 7} - \dots \infty$

(238) $\frac{1}{1\cdot 3\cdot 5} + \frac{1}{5\cdot 7\cdot 9} + \frac{1}{9\cdot 11\cdot 13} + \dots \infty$

(239) $\frac{1}{1\cdot 3\cdot 5} + \frac{1}{3\cdot 5\cdot 7} + \frac{1}{5\cdot 7\cdot 9} + \dots \infty$

(240) $\frac{1}{1\cdot 3\cdot 5} - \frac{1}{3\cdot 5\cdot 7} + \frac{1}{5\cdot 7\cdot 9} - \dots \infty$

(241) $\left(\frac{1}{1\cdot 2\cdot 3}\right)^2 + \left(\frac{1}{2\cdot 3\cdot 4}\right)^2 + \left(\frac{1}{3\cdot 4\cdot 5}\right)^2 + \dots \infty$

Reference

$$= \frac{a(b-1)}{(b-a-1)(b-a-2)} \quad \text{where } b - 2 > a > 0$$ A. 48

$$= \frac{1}{x^2}$$

$$= \frac{n}{(1+x)(1+x.\overline{n+1})}$$ F. 313

$$= \frac{1}{1-a}\left(\frac{1}{1+x} - \frac{a^n}{1+a^n x}\right)$$ F. 313

$$= \frac{1}{2}$$ T. 143

$$= \frac{\pi^2}{12}$$ 0,8224670337 E. 158

$$= \text{logh}\,\frac{4}{\epsilon}$$ 0,3862943613 C. 252

$$= \text{logh}\,2 - \frac{1}{2} = 0.1931471806$$ F. 338

$$= 0.153426 = \frac{1}{2}(1 - \text{logh}\,2)$$ 0,1534264097 H. 476

$$= \frac{\pi}{16} - \frac{1}{8}$$ 0,0713495409 0

$$= \frac{1}{12}$$ 0,08$\dot{3}$

$$= \frac{\pi}{8} - \frac{1}{3}$$ 0,0593657485 0 C. 372

$$= \frac{\pi^2}{4} - \frac{39}{16}$$ 0,0299011010 0 O. 370

Series No.

(242) $\frac{1}{1\cdot2\cdot3\cdot4}+\frac{1}{5\cdot6\cdot7\cdot8}+\frac{1}{9\cdot10\cdot11\cdot12}+\ldots\infty$

(243) $\frac{1}{1\cdot3\cdot5\cdot7}+\frac{1}{9\cdot11\cdot13\cdot15}+\frac{1}{17\cdot19\cdot21\cdot23}+\ldots\infty$

(244) $\frac{1}{2\cdot3\cdot4}-\frac{1}{4\cdot5\cdot6}+\frac{1}{6\cdot7\cdot8}-\ldots\infty$

(245) $\frac{5}{1\cdot2}-\frac{3}{2\cdot3}+\frac{9}{3\cdot4}-\frac{7}{4\cdot5}+\frac{13}{5\cdot6}-\frac{11}{6\cdot7}+\ldots n$ terms

(246) $\frac{2}{1\cdot5}-\frac{4}{5\cdot7}+\frac{8}{7\cdot17}-\frac{16}{17\cdot31}+\frac{32}{31\cdot65}-\ldots n$ terms

(247) $\frac{1}{(1\cdot3)^2}+\frac{1}{(3\cdot5)^2}+\frac{1}{(5\cdot7)^2}+\ldots\infty$

(248) $\frac{1}{2\cdot4}+\frac{1\cdot3}{2\cdot4\cdot6}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot8}+\ldots n$ terms

(249) $\frac{1}{2\cdot3\cdot4}+\frac{1}{4\cdot5\cdot6}+\frac{1}{6\cdot7\cdot8}+\ldots\infty$

(250) $\frac{1}{1\cdot2\cdot3}+\frac{1}{4\cdot5\cdot6}+\frac{1}{7\cdot8\cdot9}+\ldots\infty$

(251) $\frac{1}{2\cdot3\cdot4}+\frac{1}{6\cdot7\cdot8}+\frac{1}{10\cdot11\cdot12}+\ldots\infty$

(252) $\frac{1}{1\cdot2\cdot3\cdot4}+\frac{1}{4\cdot5\cdot6\cdot7}+\frac{1}{7\cdot8\cdot9\cdot10}+\ldots\infty$

(253) $\frac{1}{1\cdot2\cdot3}+\frac{1}{5\cdot6\cdot7}+\frac{1}{9\cdot10\cdot11}+\ldots\infty$

(254) $1-\frac{1}{2}+\frac{1\cdot3}{2\cdot4}-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}+\ldots\infty$

(255) $1+\frac{1}{2}-\frac{1\cdot1}{2\cdot4}+\frac{1\cdot1\cdot3}{2\cdot4\cdot6}-\frac{1\cdot1\cdot3\cdot5}{2\cdot4\cdot6\cdot8}+\ldots\infty$

		Reference
$= \frac{1}{4}\operatorname{logh} 2 - \frac{\pi}{24}$	0,0423871013	O. 371
$= \frac{\pi}{96(2+\sqrt{2})}$	0,009584908174	O. 371
$= \frac{1}{4}(\pi - 3)$	0,03539816340	A. 190
$= 3 - \frac{2 + (-1)^n}{n+1}$		F. 339
$= \frac{1}{3}\left\{1 + \frac{(-1)^{n+1}}{2^{n+1} + (-1)^{n+1}}\right\}$		F. 339
$= \frac{\pi^2 - 8}{16}$	0,1168502753	
$= \frac{1}{2} - \frac{1\cdot 3\cdot 5\ldots(2n+1)}{2\cdot 4\cdot 6\ldots(2n+2)}$		F. 333
$= \frac{3}{4} - \operatorname{logh} 2$	0,05685281940	T. 144
$= \frac{1}{4}\left(\frac{\pi}{\sqrt{3}} - \operatorname{logh} 3\right)$	0,1787967688	T. 144
$= \frac{\pi}{8} - \frac{1}{2}\operatorname{logh} 2$	0,04612549150	T. 144
$= \frac{1}{6}\left(1 + \frac{\pi}{2\sqrt{3}}\right) - \frac{1}{4}\operatorname{logh} 3$	0,04316354140	T. 144
$= \frac{1}{4}\operatorname{logh} 2$	0,1732867952	C. 252
$= \frac{1}{\sqrt{2}}$		
$= \sqrt{2}$		

Series No.

(256) $\frac{1}{3} + \frac{2}{3\cdot 5} + \frac{3}{3\cdot 5\cdot 7} + \frac{4}{3\cdot 5\cdot 7\cdot 9} + \ldots \infty$

(257) $\frac{1}{3} + \frac{2}{3\cdot 5} + \frac{3}{3\cdot 5\cdot 7} + \frac{4}{3\cdot 5\cdot 7\cdot 9} + \ldots$ n terms

(258) $1 + \frac{1}{3} + \frac{1\cdot 2}{3\cdot 5} + \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7} + \ldots \infty$

(259) $1 + \frac{1}{3} + \frac{1\cdot 2}{3\cdot 5} + \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7} + \ldots \infty$

$$= 1 + \frac{2}{3}\cdot\frac{1}{2} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{1}{2^2} + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot\frac{1}{2^3} + \ldots$$

(260) $1 + \frac{1}{2\cdot 3} + \frac{1\cdot 3}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7} + \ldots \infty$

(261) $1 + \frac{1}{3}\cdot\frac{1}{2} + \frac{1\cdot 2}{3\cdot 5}\cdot\frac{1}{2^2} + \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\cdot\frac{1}{2^3} + \ldots \infty$

(262) $\frac{1}{2} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{3} + \ldots \infty$

(263) $\frac{1}{2\cdot 5} + \frac{1}{8\cdot 11} + \frac{1}{14\cdot 17} + \ldots \infty$

(264) $1 - \frac{1}{7} + \frac{1}{9} - \frac{1}{15} + \frac{1}{17} - \ldots \infty$

(265) $\frac{1}{1\cdot 5} + \frac{1}{2\cdot 7} + \frac{1}{3\cdot 9} + \ldots \infty$

(266) $\frac{1}{1\cdot 5} + \frac{1}{9\cdot 13} + \frac{1}{17\cdot 21} + \frac{1}{25\cdot 29} + \ldots \infty$

(267) $1 - \frac{2}{3}\cdot\frac{1}{2} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{1}{2^2} - \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot\frac{1}{2^3} + \ldots \infty$

(268) $\sum_{1}^{\infty} \frac{2}{(t + n - 1)(t + n)(t + n + 1)}$

		Reference
$= \frac{1}{2}$		C. 225
$= \frac{1}{2}\left\{1 - \frac{1}{3\cdot 5\cdot 7\ldots(2n+1)}\right\}$		A. 431
$= \frac{\pi}{2}$		Y. 505
$= \frac{\pi}{2}$		A. 197
$= \frac{\pi}{2}$		A. 184
$= \frac{2\pi}{3\sqrt{3}}$	1,209199576	Y. 505
$= 2 \text{ logh } 2$	1,3862943613	1Z. 136
$= \frac{1}{9}\left(\frac{\pi}{\sqrt{3}} - \text{logh } 2\right)$	0,1245169092	1Z. 164
$= \frac{\pi}{8}(1 + \sqrt{2})$	0,948059449	1Z. 165
$= \frac{8}{9} - \frac{2}{3} \text{ logh } 2$	0,4267907685	1Z. 165
$= \frac{\sqrt{2}}{32}\{\pi + \text{logh}(3 + 2\sqrt{2})\}$	0,216743246 8	1Z. 165
$= \frac{2}{\sqrt{3}} \text{ logh } \frac{1+\sqrt{3}}{\sqrt{2}}$		A. 197
$= \frac{1}{t(t+1)}$	idem	A. 52

Séries L(1,χ)
Dirichlet.

Series No.

(269) $\sum_{1}^{\infty} \frac{3}{(t+n-1)(t+n)(t+n+1)(t+n+2)}$

(270) $\frac{1}{2} + \frac{1}{2\cdot 3\cdot 2^3} + \frac{1\cdot 3}{2\cdot 4\cdot 5\cdot 2^5} + \dots \infty$

(271) $\frac{1}{1\cdot 2^2\cdot 3} + \frac{1}{5\cdot 6^2\cdot 7} + \frac{1}{9\cdot 10^2\cdot 11} + \dots \infty$

(272) $\frac{1}{1^2.2} + \frac{1}{2^2.3} + \frac{1}{3^2.4} + \dots \infty$

(273) $1 - \frac{1}{3\cdot 3} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \dots \infty$

(274) $1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2\cdot 4}\right)^2 + \left(\frac{1\cdot 3}{2\cdot 4\cdot 6}\right)^2 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\right)^2 + \dots \infty$

(275) $1 + \frac{1}{3}\left(\frac{1}{2}\right) + \frac{1\cdot 2}{3\cdot 5}\left(\frac{1}{2}\right)^2 + \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\left(\frac{1}{2}\right)^3 + \dots \infty$

(276) $1 + \frac{1}{2\cdot 3} + \frac{1}{3}\cdot\frac{1\cdot 2}{3\cdot 5} + \frac{1}{4}\cdot\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7} + \dots \infty$

(277) $1 + \frac{1}{2}\cdot\frac{1}{3}\left(\frac{1}{2}\right) + \frac{1}{3}\cdot\frac{1\cdot 2}{3\cdot 5}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\left(\frac{1}{2}\right)^3 + \dots \infty$

(278) $1 + \frac{3}{6} + \frac{3\cdot 6}{6\cdot 10} + \frac{3\cdot 6\cdot 9}{6\cdot 10\cdot 14} + \dots \infty$

(279) $\frac{1}{1\cdot 2\cdot 3} + \frac{x}{2\cdot 3\cdot 4} + \frac{x^2}{3\cdot 4\cdot 5} + \dots \infty$

(280) $\frac{1}{1\cdot 2\cdot 3} + \frac{x}{3\cdot 4\cdot 5} + \frac{x^2}{5\cdot 6\cdot 7} + \dots \infty$

(281) $\frac{1}{1\cdot 2\cdot 3} - \frac{x}{3\cdot 4\cdot 5} + \frac{x^2}{5\cdot 6\cdot 7} - \dots \infty$

Reference

$$= \frac{1}{t(t+1)(t+2)}$$ A. 52

$$= \frac{\pi}{6}$$ H. 468

$$= \frac{\pi}{8}\left(1 - \frac{\pi}{4}\right)$$ D. 495

$$= \frac{\pi^2 - 6}{6}$$ C. 372

$$= \frac{\pi}{2\sqrt{3}}$$ H. 476

$$= \frac{\Gamma(2)}{[\Gamma(3/2)]^2} = \frac{4}{\pi}$$ A. 190

$$= \frac{2\pi}{3\sqrt{3}}$$ L. 236

$$= \frac{\pi^2}{8}$$ L. 237

$$= \frac{\pi^2}{9}$$ L. 237

$$= \frac{4\pi}{3\sqrt{3}}$$ L. 237

$$= \frac{3}{4x} - \frac{1}{2x^2} + \frac{(1-x)^2}{2x^3} \operatorname{logh} \frac{1}{1-x} \quad \text{where } x^2 < 1$$ T. 125

$$= \frac{1}{4x}\left\{\frac{1+x}{\sqrt{x}} \operatorname{logh} \frac{1+\sqrt{x}}{1-\sqrt{x}} + 2 \operatorname{logh}(1-x) - 2\right\} \quad \text{where } 0 < x < 1$$ T. 125

$$= \frac{1}{2x}\left\{1 - \operatorname{logh}(1+x) - \frac{1-x}{\sqrt{x}} \tan^{-1} x\right\} \quad \text{where } 0 < x \leqslant 1$$ T. 125

Series No.

XII. Simple Factorials

(282) $\frac{1}{3!} + \frac{5}{4!} + \frac{11}{5!} + \ldots n$ terms

(283) $\frac{2}{1!} + \frac{12}{2!} + \frac{28}{3!} + \frac{50}{4!} + \frac{78}{5!} + \ldots \infty$

(284) $\frac{1}{n!} + \frac{2!}{(n+1)!} + \frac{3!}{(n+2)!} + \ldots \infty$

(285) $\frac{0!}{n!} + \frac{1!}{(n+1)!} + \frac{2!}{(n+2)!} + \ldots \infty$

(286) $\sum_0^\infty \frac{(2n)!}{2^{2n}n!n!} \cdot \frac{1}{x+n}$

(287) $\sum_{n=0}^\infty \frac{(-1)^n a(a-1)(a-2)\ldots(a-n)}{n!} \cdot \frac{1}{x+n}$

(288) $\sum_1^n n(n!)$

(289) $\sum_1^\infty (-1)^{n+1} \frac{1}{n(n!)}$

(290) $\frac{1}{m} - \frac{n}{1!} \frac{1}{m+1} + \frac{n(n-1)}{2!} \cdot \frac{1}{m+2} - \ldots$

(291) $\frac{1}{3!} + \frac{5}{4!} + \frac{11}{5!} + \ldots \frac{n^2+n-1}{(n+2)!}$

XIII. Other Power Series (Bernoulli's and Euler's Numbers)

(292) $\underset{n=\infty}{\mathrm{Lt}} \left[\frac{1^2}{n^3+1^3} + \frac{2^2}{n^3+2^3} + \frac{3^2}{n^3+3^3} + \ldots + \frac{n^2}{n^3+n^3}\right]$

(293) $\underset{n=\infty}{\mathrm{Lt}} \left[\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{n+n}\right]$

Reference

$$= \frac{1}{2} - \frac{n+1}{(n+2)!}$$ F. 333

$$= 5\epsilon + 2$$ F. 334

$$= \frac{1}{(n-2)(n-1)!}$$ F. 338

$$= \frac{1}{(n-1)(n-1)!}$$ AC. 63

$$= \frac{\Gamma(x)\Gamma(\frac{1}{2})}{\Gamma(x+\frac{1}{2})}$$ Q. 259

$$= \frac{\Gamma(x)\Gamma(a+1)}{\Gamma(x+a)}$$ Q. 260

$$= (n+1)! - 1$$

$$= 0.7965996 = \int_0^1 \frac{1-e^{-x}}{x}\,dx$$ *

$$= \frac{(m-1)!}{(n+1)(n+2)\ldots(n+m)}$$ C. 211

$$= \frac{1}{2} - \frac{1}{(n+2)(n!)}$$ Hello!

$$= \frac{1}{3}\,\text{logh}\,2$$ 1Z. 324

$$= \text{logh}\,2$$ 1Z. 326

Series No.

(294) $\underset{n=\infty}{\text{Lt}} \left[\frac{n}{n^2 + 1^2} + \frac{n}{n^2 + 2^2} + \ldots + \frac{n}{n^2 + n^2}\right]$

(295) $\underset{n=\infty}{\text{Lt}} \left[\frac{1}{\sqrt{2n - 1^2}} + \frac{1}{\sqrt{4n - 2^2}} + \frac{1}{\sqrt{6n - 3^2}} + \ldots + \frac{1}{\sqrt{2n^2 - n^2}}\right]$

(296) $\underset{n=\infty}{\text{Lt}} \left[\frac{(n - m)^{1/3}}{n} + \frac{(2^2n - m)^{1/3}}{2n} + \frac{(3^2n - m)^{1/3}}{3n} + \ldots + \frac{(n^3 - m)^{1/3}}{n^2}\right]$

(297) $\underset{n=\infty}{\text{Lt}} \left[\frac{\sqrt{n - 1}}{n} + \frac{\sqrt{2n - 1}}{2n} + \frac{\sqrt{3n - 1}}{3n} + \ldots + \frac{\sqrt{n^2 - 1}}{n^2}\right]$

(298) $\underset{n=\infty}{\text{Lt}} \left[\frac{1}{\sqrt{2a^2n - 1}} + \frac{1}{\sqrt{4a^2n - 1}} + \ldots + \frac{1}{\sqrt{2a^2n^2 - 1}}\right]$

(299) $\underset{n=\infty}{\text{Lt}} \left[\frac{n^2}{(n^2 + 1^2)^{3/2}} + \frac{n^2}{(n^2 + 2^2)^{3/2}} + \ldots + \frac{n^2}{\{n^2 + (n - 1)^2\}^{3/2}}\right]$

(300) $\underset{n=\infty}{\text{Lt}} \left[\frac{\sqrt{n - a}}{n - c} + \frac{\sqrt{2n - a}}{2n - c} + \ldots + \frac{\sqrt{n^2 - a}}{n^2 - c}\right]$

(301) $\frac{1}{n + 1} + \frac{2}{n^2 + 1} + \frac{4}{n^4 + 1} + \ldots \infty$

(302) $\sum_{1}^{\infty} \frac{n - 1}{n!}$

(302); $1/2 + 1/3 + 1/8 + 1/30 + 1/144 + 1/840 + 1/5760 + 1/45360 + 1/403200 + 1/3991680 + \frac{(m-1)}{m!} + \ldots$

Reference

$= \frac{\pi}{4}$ 1Z. 326

$= \frac{\pi}{2}$ 1Z. 326

$= \frac{3}{2}$ 1Z. 326

$= 2$ 1Z. 326

$= \frac{\sqrt{2}}{a}$ 1Z. 326

$= \frac{1}{\sqrt{2}}$ 1Z. 326

$= 2$ 1Z. 354

$= \frac{1}{n-1}$ where $|n| > 1$ A. 66

$= 1$ A. 52

Series No.

(303) $\left(n+\frac{1}{2}\right)\text{logh}\, n - n + \frac{1}{2}\text{logh}\,(2\pi) + \frac{1}{12n} - \frac{1}{360n^3} - \ldots \infty$

(304) $n^{n+1/2}\sqrt{2\pi}\epsilon^{-n}\left\{1 + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \ldots \infty\right\}$

n large

On the series Nos. (305) to (330), see No. (1130) for values of α, β, etc.; see No. (330) for general note covering Nos. (305) through (329).

(305) $1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \ldots \infty$

(306) $1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \ldots \infty$

(307) $1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \ldots \infty$

(308)† $1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \ldots \infty$

(309) $1 + \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{8^n} + \ldots \infty$

(310) $1 - \frac{1}{2^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{8^n} + \ldots \infty$

(311) $1 + \frac{1}{2^n} - \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{8^n} - \ldots \infty$

† $u_1 = \frac{\pi}{4}$
$u_2 = 0.91596\,56\ldots$
$u_4 = 0.98894\,455\ldots$
$u_6 = 0.99868\,522.$
A table of *n* from 1 to 38 to 18 decimal places is given by Glaisher, *Messenger of Mathematics*, **42**; 49, 1913.

Reference

$= \operatorname{logh}(n!)$ G. 140

$= n! = \sqrt{2\pi n}\left(\frac{n}{\epsilon}\right)^n$ G. 140

$= \mathcal{S}_n, \quad \mathcal{S}_{2n} = \frac{\frac{1}{2}(2\pi)^{2n}B_n}{(2n)!}$ AC. 42

$= s_n, \quad s_{2n} = \frac{\pi^{2n}\beta_n}{2\cdot(2n)!}$ AC. 42

$= U_n, \quad U_{2n} = \frac{\pi^{2n}\alpha_n}{4\cdot(2n)!}$ AC. 42

$= u_n, \quad u_{2n+1} = \frac{\left(\frac{\pi}{2}\right)^{2n+1}E_n^*}{2(2n)!}$ AC. 42

$= G_n, \quad G_{2n} = \frac{\frac{2}{3}(\frac{2}{3}\pi)^{2n}\gamma_n}{(2n)!}$ AC. 42

$= g_n, \quad \sqrt{3}g_{2n+1} = \frac{(\frac{2}{3}\pi)^{2n+1}I_n}{(2n)!}$ AC. 42

$= j_n, \quad \sqrt{3}j_{2n+1} = \frac{(\frac{1}{3}\pi)^{2n+1}J_n}{(2n)!}$ AC. 42

T. 140

Series No.

(312) $1 - \frac{1}{2^n} - \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{8^n} - \ldots \infty$

(313) $1 + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \ldots \infty$

(314) $1 - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \ldots \infty$

(315) $1 + \frac{1}{5^n} - \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} - \ldots \infty$

(316) $1 - \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} - \ldots \infty$

(317) $1 + \frac{1}{3^n} - \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} + \frac{1}{11^n} - \ldots \infty$

(318) $1 - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} - \ldots \infty$

(319) $1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \ldots \infty$

(320) $1 - \frac{1}{2^{2n}} + \frac{1}{3^{2n}} - \frac{1}{4^{2n}} + \ldots \infty$

(321) $1 - \frac{1}{2^{2n+1}} + \frac{1}{4^{2n+1}} - \frac{1}{5^{2n+1}} + \ldots \infty$

(322) $1 + \frac{1}{2^{2n}} - \frac{2}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} - \frac{2}{6^{2n}} + \ldots \infty$

(323) $1 + \frac{1}{2^{2n+1}} - \frac{1}{4^{2n+1}} - \frac{1}{5^{2n+1}} + \ldots \infty$

(324) $1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \ldots \infty$

Reference

$$= v_n, \quad v_{2n} = \frac{\frac{4}{3}(\frac{1}{3}\pi)^{2n}\eta_n}{(2n)!}$$ AC. 42

$$= W_n, \quad W_{2n} = \frac{2(\frac{1}{3}\pi)^{2n}\theta_n}{(2n)!}$$ AC. 42

$$= h_n, \quad \sqrt{3}h_{2n+1} = \frac{(\frac{1}{3}\pi)^{2n+1}H_n}{(2n)!}$$ AC. 42

$$= r_n, \quad r_{2n+1} = \frac{2(\frac{1}{6}\pi)^{2n+1}R_n}{(2n)!}$$ AC. 42

$$= t_n, \quad \sqrt{3}t_{2n} = \frac{6(\frac{1}{6}\pi)^{2n}T_n}{(2n-1)!}$$ AC. 42

$$= p_n, \quad p_{2n+1} = \frac{\sqrt{2}(\frac{1}{4}\pi)^{2n+1}P_n}{(2n)!}$$ AC. 42

$$= q_n, \quad q_{2n} = \frac{\sqrt{2}(\frac{1}{4}\pi)^{2n}Q_n}{(2n-1)!}$$ AC. 42

$$= \frac{(2\pi)^{2n}}{2(2n)!} B_n$$ AE. 3

$$= \frac{2^{2n-1}\pi^{2n}}{(2n)!}\left(1 - \frac{1}{2^{2n-1}}\right) B_n$$ AE. 26

$$= \frac{(-1)^{n+1}(2\pi)^{2n+1}}{\sqrt{3}(2n)!} B_{2n+1}\left(\frac{1}{3}\right)$$ AE. 34

$$= \frac{(-1)^{n}(2\pi)^{2n}}{(2n-1)!} A_{2n}\left(\frac{1}{3}\right)$$ AE. 34

$$= \frac{(-1)^{n+1}(2\pi)^{2n+1}}{\sqrt{3}(2n)!} B_{2n+1}\left(\frac{1}{6}\right)$$ AE. 40

$$= \frac{(-1)^{n+1}2^{2n}\pi^{2n+1}}{(2n)!} B_{2n+1}\left(\frac{1}{4}\right)$$ AE. 30

$$= \frac{\pi^{2n+1}}{2^{2n+2}(2n)!} E_n^*$$ AE. 30

Series No.

(325) $\frac{1}{2^{2n}} - \frac{1}{4^{2n}} + \frac{1}{6^{2n}} - \frac{1}{8^{2n}} + \dots \infty$

(326) $1 + \frac{1}{3^{2n+1}} - \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \frac{1}{9^{2n+1}} + \dots \infty$

(327) $1 - \frac{1}{3^{2n}} - \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{9^{2n}} - \dots \infty$

(328) $1 + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} - \frac{1}{11^{2n+1}} + \dots \infty$

(329) $1 - \frac{1}{5^{2n}} - \frac{1}{7^{2n}} + \frac{1}{11^{2n}} + \dots \infty$

(330) General note on Nos. (305) through (329):

(a) The values of Bernoulli's and Euler's numbers are given in Nos. (1129) and (1131).

(b) The values of $B_n(x)$ and $A_n(x)$, etc., are given in Nos. (1134) to (1146).

(c) The coefficients $\mathcal{S}$, etc., in Nos. (305) to (312) are given in No. (1130). See also No. (1101).

(d) The values of p_{2n+1}, q_{2n}, r_{2n+1}, and t_{2n} are given in the table opposite for values of $n = 0$ to 4.

(e) The summation of No. (305) to 16 places of decimals is given in No. (1133).

(f) Some of these series are derived from No. (546), etc., giving θ appropriate values.

(g) Between Nos. (305) to (318) and (319) to (329) there is some duplication, as the series are collected from different sources, but the results are compatible.

Reference

$$= \frac{(-1)^n(2\pi)^{2n}}{2(2n+1)!}\left\{B_{2n}\left(\frac{1}{4}\right) + (-1)^{n-1}\frac{B_n}{2n}\right\}$$ AE. 30

$$= p_{2n+1}$$ AE. 59

$$= q_{2n}$$ AE. 64

$$= r_{2n+1}$$ AE. 69

$$= t_{2n}$$ AE. 74

n	p_{2n+1}	q_{2n}	r_{2n+1}	t_{2n}
0	$\frac{\pi}{2\sqrt{2}}$	—	$\frac{\pi}{3}$	—
1	$\frac{3\pi^3}{\sqrt{2}\cdot 2^6}$	$\frac{\pi^2}{\sqrt{2}\cdot 2^3}$	$\frac{7\pi^3}{6^3}$	$\frac{\pi^2}{2\cdot 3\sqrt{3}}$
2	$\frac{19\pi^5}{\sqrt{2}\cdot 2^{12}}$	$\frac{11\pi^4}{3\sqrt{2}\cdot 2^8}$	$\frac{5\cdot 61\pi^5}{2^7\cdot 3^6}$	$\frac{23\pi^4}{2^4\cdot 3^4\sqrt{3}}$
3	$\frac{307\pi^7}{5\sqrt{2}\cdot 2^{17}}$	$\frac{19^2\cdot\pi^6}{3\cdot 5\sqrt{2}\cdot 2^{14}}$	$\frac{61\cdot 547\pi^7}{2^{10}\cdot 3^9\cdot 5}$	$\frac{41^2\pi^6}{2^8\cdot 3^6\cdot 5\sqrt{3}}$
4	—	$\frac{24611\pi^8}{2^{19}\cdot 3^2\cdot 5\cdot 7\sqrt{2}}$	—	$\frac{11\cdot 13\cdot 1801\pi^8}{2^{11}\cdot 3^9\cdot 5\cdot 7\sqrt{3}}$
Ref.:	AE. 63	AE. 66	AE. 73	AE. 77

Series No.

(331) $\frac{(-1)^{n+1}(2\pi)^{2n+1}}{2\cdot(2n)!}\left\{B_{2n+1}\left(\frac{1}{8}\right)-\frac{1}{2^{2n+1}}B_{2n+1}\left(\frac{1}{4}\right)\right\}$

(332) $\frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n-1)!}\left\{A_{2n}\left(\frac{1}{8}\right)-\frac{1}{2^{2n}}A_{2n}\left(\frac{1}{4}\right)\right\}$

(333) $\frac{(-1)^{n+1}(2\pi)^{2n+1}}{2\cdot(2n)!}\left\{B_{2n+1}\left(\frac{5}{12}\right)+\frac{1}{2^{2n+1}}B_{2n+1}\left(\frac{1}{6}\right)\right.$

$$\left.-\frac{1}{3^{2n+1}}B_{2n+1}\left(\frac{1}{4}\right)\right\}$$

(334) $\frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n-1)!}\left\{A_{2n}\left(\frac{1}{12}\right)-\frac{1}{2^{2n}}A_{2n}\left(\frac{1}{6}\right)\right\}$

In the above four series see also No. (1146) for an amplification of the coefficients A_n, B_n, etc.

(335) If $\sum_{1}^{\infty}\frac{1}{n^s}$

$\frac{1}{1^s}\quad\frac{1}{3^s}\quad\frac{1}{5^s}+\ldots\infty$

$\frac{1}{1^s}\quad\frac{1}{5^s}\quad\frac{1}{7^s}+\ldots\infty$

Generally

$1+\sum' n^{-s}$

The dash sign indicates that only those values of n (greater than p) which are prime to $2\cdot3\cdot4\ldots p$ occur in the summation

(336) $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\infty$

(337) $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots\infty$

(338) $1+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\ldots\infty$

Reference

$$= \frac{1}{\sqrt{2}} p_{2n+1}$$ AE. 59

$$= \frac{1}{\sqrt{2}} q_{2n}$$ AE. 64

$$= \frac{1}{2} r_{2n+1}$$ AE. 73

$$= \frac{\sqrt{3}}{2} t_{2n}$$ AE. 74

$= \zeta(s)$ 2, 3, 5...p—are prime numbers in order

$$= \zeta(s)(1 - 2^{-s})$$

$$= \zeta(s)(1 - 2^{-s})(1 - 3^{-s})$$

$$= \zeta(s)(1 - 2^{-s})\ldots(1 - p^{-s})$$ Q. 272

$$= \frac{\pi^2}{6}$$ E. 154

$$= \frac{\pi^2}{12}$$ E. 158

$$= \frac{\pi^2}{9}$$ 27 P. 281

↓ Encyclopédie Brittanique 11e éd

Series No.

(339) $1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \infty$

(340) $1 + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} + \ldots \infty$

(341) $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots \infty$

(342) $1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots \infty$

(343) $1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots \infty$

(344) $1 - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} - \ldots \infty$

(345) $1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \ldots \infty$

(346) $\frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{17^2} + \frac{1}{19^2} + \frac{1}{29^2} + \frac{1}{31^2} + \ldots \infty$

(347) $\frac{1}{2^2} - \frac{2}{3^2} + \frac{3}{4^2} - \frac{4}{5^2} + \ldots \infty$

(348) $\frac{1}{1^2 . 2} + \frac{1}{2^2 . 3} + \frac{1}{3^2 . 4} + \ldots \infty$

(349) $\frac{1}{3^4} + \frac{1 + 2}{5^4} + \frac{1 + 2 + 3}{7^4} + \frac{1 + 2 + 3 + 4}{9^4} + \ldots \infty$

(350) $1^2 - \frac{2^2}{5} + \frac{3^2}{5^2} - \frac{4^2}{5^3} + \ldots \infty$

(351) $\left(\frac{1}{2} - \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} + \ldots \infty\right)$

$$+ \left(\frac{1}{3} - \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^3} - \ldots \infty\right)$$

	Reference
$= \frac{\pi^2}{8}$	E. 155
$= \frac{3\pi^3\sqrt{2}}{128}$	A. 364
$= \frac{\pi^3}{32}$	Y. 501
$= \frac{\pi^4}{96}$	E. 155
$= \frac{\pi^4}{90}$	E. 154
$= \frac{11\pi^4\sqrt{2}}{1536}$	A. 364
$= \frac{5\pi^5}{1536}$	Y. 501
$= \frac{\pi^2(2 - \sqrt{3})}{36}$	A. 528
$= \frac{\pi^2}{12} - \text{logh } 2$	
$= \frac{\pi^2 - 6}{6}$	
$= \frac{\pi^2}{64}\left(1 - \frac{\pi^2}{12}\right)$	E. 158
$= \frac{25}{54}$	F. 332
$= \frac{\pi}{4}$	H. 462

Series No.

(352) $1 - \frac{1}{2^2} + \frac{1}{2^4} - \frac{1}{2^6} + \ldots \infty$

(353) $1 - \frac{1}{3^2} + \frac{1}{3^4} - \frac{1}{3^6} + \ldots \infty$

(354) $1 - \frac{1}{4^2} + \frac{1}{4^4} - \frac{1}{4^6} + \ldots \infty$

(355) $1 - \frac{1}{5^2} + \frac{1}{5^3} - \frac{1}{5^6} + \ldots \infty$

(356) $\sum_1^n \frac{1}{n^2}$

(357) $\sum_1^n \frac{1}{n^3}$

(358) $1 + \frac{1}{5^2} + \frac{1}{9^2} + \ldots$ n terms

(359) $\sum_1^\infty \frac{1}{n \cdot 2^n}$

(360) $\frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \ldots \infty$ can be summed in five cases only:

(a) $x = 1, \quad \sum_1^\infty \frac{x^n}{n^2}$

(b) $x = -1, \quad \sum_1^\infty \frac{x^n}{n^2}$

Reference

$$= \frac{4}{5}$$ T. 144

$$= \frac{9}{10}$$ T. 144

$$= \frac{16}{17}$$ T. 144

$$= \frac{25}{26}$$ T. 144

$$= \frac{\pi^2}{6} - \frac{b_1}{n+1} - \frac{b_2}{(n+1)(n+2)} - \frac{b_3}{(n+1)(n+2)(n+3)} - \cdots$$

where $b_k = \frac{(k-1)!}{k}$ T. 27

$$= K - \frac{C_2}{(n+1)(n+2)} - \frac{C_3}{(n+1)(n+2)(n+3)} - \cdots$$

where $K = 1.2020569 = \sum_1^\infty \frac{1}{k^3}$, see (1133),

and $C_k = \frac{(k-1)!}{k}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{k-1}\right)$ T. 27

$$= 1.0787 - \frac{1}{16}\left[\frac{1}{n+\frac{1}{4}} + \frac{1}{2(n+\frac{1}{4})} + \frac{1}{6(n+\frac{1}{4})^3} + \frac{1}{30(n+\frac{1}{4})^5} + \cdots\right]$$ AC. 63 (Glaisher 1876)

$$= \operatorname{logh} 2$$

$$= \frac{\pi^2}{6}$$

$$= \frac{\pi^2}{12}$$

Series No.

(c) $x = \frac{1}{2}, \quad \sum_1^\infty \frac{x^n}{n^2}$

(d) $x = 2 \sin \frac{\pi}{10}, \quad \sum_1^\infty \frac{x^n}{n^2}$

(e) $x = \left(2 \sin \frac{\pi}{10}\right)^2, \quad \sum_1^\infty \frac{x^n}{n^2}$

(361) $\sum_1^\infty \frac{n^r}{n!}$

(362) $\sum_1^\infty \frac{1}{n^2} \cdot \frac{1}{2^n} = \int_0^1 \operatorname{logh}\left(\frac{1}{x}\right) \frac{dx}{2-x}$

(363) $\sum_1^n \left(1 + \frac{1}{x^n}\right)^2$

(364) $\sum_1^\infty \frac{n}{(4n^2-1)^2}$

(365) $\sum_1^\infty \frac{(-1)^n}{n(n+1)^2} = \int_0^1 \operatorname{logh} x \operatorname{logh}(1+x)dx$

(366) $\sum_1^\infty (-1)^{n-1} \frac{1}{4n^2-1}$

(367) $\sum_1^\infty \frac{1}{(2n+1)^2-1}$

(368) $\sum_1^\infty (-1)^{n-1} \frac{1}{(2n+1)^2-1}$

(369) $\sum_1^\infty \frac{1}{n^2-1}$

2

Reference

$$= \frac{\pi^2}{12} - \frac{1}{2}\left(\text{logh}\,\frac{1}{2}\right)^2$$

$$= \frac{\pi^2}{10} - \left(\text{logh}\,2\sin\frac{\pi}{10}\right)^2$$

$$= \frac{\pi^2}{15} - \text{logh}\left(2\sin\frac{\pi}{10}\right)^2$$ 2Z. 286

Nombres de Bell

$$= \mathcal{S}_r$$

$\mathcal{S}_1 = \epsilon \quad \mathcal{S}_2 = 2\epsilon \quad \mathcal{S}_3 = 5\epsilon \quad \mathcal{S}_4 = 15\epsilon$

$\mathcal{S}_5 = 52\epsilon \quad \mathcal{S}_6 = 203\epsilon \quad \mathcal{S}_7 = 877\epsilon \quad \mathcal{S}_8 = 4140\epsilon$ A. 197

$$= \frac{\pi^2}{12} - \frac{1}{2}(\text{logh}\,2)^2$$ A. 520

$$= n + \frac{2(1 - x^{-n})}{x - 1} + \frac{1 - x^{-2n}}{x^2 - 1}$$

$$= \frac{1}{8}$$ A. 52

$$= \sum (-1)^{n-1} \int_0^1 \frac{x^n}{n}\,\text{logh}\,x\,dx = 2 - 2\,\text{logh}\,2 - \frac{\pi^2}{12}$$ A. 496

$$= \frac{\pi}{4} - \frac{1}{2}$$

$$= \frac{1}{4}$$

$$= \frac{1}{2}\,\text{logh}\,2 - \frac{1}{4}$$

$$= \frac{3}{4}$$

Series No.

(370) $\sum_{1}^{\infty} \frac{x^n}{n+a}$

(371) $\sum_{2}^{n} \frac{1}{n^2-1}$

(372) $\sum_{1}^{\infty} \frac{x^{n-1}}{(1-x^n)(1-x^{n+1})}$

(373) $\sum_{1}^{\infty} \frac{1}{(4n^2-1)^r}$

(374) $\sum_{1}^{\infty} \frac{1}{n(4n^2-1)}$

(375) $\sum_{1}^{\infty} \frac{1}{n(9n^2-1)}$

(376) $\sum_{1}^{\infty} \frac{1}{n(36n^2-1)}$

(377) $1 + x^2 - \frac{x^3}{3} + \frac{5}{6}x^4 - \frac{3}{4}x^5 \ldots \infty$

(378) $\sum_{n=1}^{n=\infty} \left[\frac{1}{(3n-1)^4} + \frac{1}{(3n+1)^4}\right]$

(379) $\frac{1}{n+1} + \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \ldots \infty$

(380) $1 - \frac{n^2}{1} + \frac{n^2(n^2-1^2)}{1^2 \cdot 2^2} - \frac{n^2(n^2-1^2)(n^2-2^2)}{1^2 \cdot 2^2 \cdot 3^2}$

... to $n+1$ terms

(381) $\frac{x}{3} + \frac{1 \cdot 4}{3 \cdot 6} 2^3 \cdot x^2 + \frac{1 \cdot 4 \cdot 7}{3 \cdot 6 \cdot 9} \cdot 3^3 \cdot x^3 + \ldots \infty$

Reference

$$= -x^{-a}\left\{\frac{x}{1} + \frac{x^2}{2} + \ldots \frac{x^a}{a} + \operatorname{logh}(1 - x)\right\} \quad \text{where } x < 1$$ C. 246

$$= \frac{3}{4} - \frac{1}{2n} - \frac{1}{2(n+1)}$$

$$= \frac{1}{(1 - x)^2} \quad \text{where } |x| < 1$$ Q. 59

$$= \frac{1}{x(1 - x)^2} \quad \text{where } |x| > 1$$

$$= \mathcal{S}_r$$

$$\mathcal{S}_1 = \frac{1}{2} \quad \mathcal{S}_2 = \frac{\pi^2 - 8}{16} \quad \mathcal{S}_3 = \frac{32 - 3\pi^2}{64}$$

$$\mathcal{S}_4 = \frac{\pi^4 + 30\pi^2 - 384}{768}$$ T. 141

$$= 2 \operatorname{logh} 2 - 1$$ T. 142

$$= \frac{3}{2}(\operatorname{logh} 3 - 1)$$ T. 142

$$= -3 + \frac{3}{2}\operatorname{logh} 3 + 2 \operatorname{logh} 2$$ T. 142

ref EIS

$$= (1 + x)^x$$ Y. 107

$$= \frac{8\pi^4}{729} - 1$$ ≃ 2/29 E. 190

$$= \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots \infty$$ F. 197

$$= 0$$ F. 338

$$= \frac{1}{27}\cdot\frac{x}{(1 - x)^{10/3}}(x^2 + 18x + 9)$$

Series No.

(382) $$\sum_{n=1}^{n=\infty} \frac{1}{\{(2n)^2 - (2m-1)^2\}^2}$$

(383) $$\sum_{n=1}^{n=\infty} \frac{1}{\{(2n-1)^2 - (2m)^2\}^2}$$

(384) $$\underset{n=\infty}{\mathrm{Lt}} \frac{\left(\frac{1}{2n}\right)^p + \left(\frac{2}{2n}\right)^p + \left(\frac{3}{2n}\right)^p + \ldots 2n \text{ terms}}{\left(\frac{1}{2} + \frac{1}{2n}\right)^p + \left(\frac{1}{2} + \frac{2}{2n}\right)^p + \ldots n \text{ terms}}$$

(385) $$\sum_{1}^{\infty} \left\{\frac{1\cdot 3\cdot 5\ldots 2n-1}{2\cdot 4\cdot 6\ldots 2n}\right\}^2 \frac{1}{2n+r}$$

$$\mathcal{S}_0 = 2\,\mathrm{logh}\,2 - \frac{4}{\pi}\omega_2$$

$$\mathcal{S}_1 = \frac{4}{\pi}\omega_2 - 1$$

$$\mathcal{S}_2 = \frac{2}{\pi} - \frac{1}{2}$$

$$\mathcal{S}_3 = \frac{1}{2\pi}(2\omega_2 + 1) - \frac{1}{3}$$

$$\mathcal{S}_4 = \frac{10}{9\pi} - \frac{1}{4}$$

$$\mathcal{S}_5 = \frac{1}{32\pi}(18\omega_2 + 13) - \frac{1}{5}$$

$$\mathcal{S}_6 = \frac{178}{225\pi} - \frac{1}{6}$$

$$\mathcal{S}_7 = \frac{1}{128\pi}(50\omega_2 + 43) - \frac{1}{7}$$

$$\omega_2 = \sum_{0}^{\infty} (-1)^{k-1} \frac{1}{(2k+1)^2}$$

$$= 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots \infty \doteq$$

$$= 0.9159656$$ $\longrightarrow$ CATALAN

When r is a negative integer the value of $n = r/2$ is to be excluded in the summation.

Reference

$$= \frac{\pi^2}{16(2m-1)^2} - \frac{1}{2(2m-1)^4}$$ C. 373

$$= \frac{\pi^2}{64m^2}$$ C. 373

$$= \frac{1}{1-\left(\frac{1}{2}\right)^{p+1}}$$ J. 74

$$= \mathcal{S}_r$$ T. 142

$$\mathcal{S}_{-1} = 1 - \frac{2}{\pi}$$

$$\mathcal{S}_{-2} = \frac{1}{2}\operatorname{logh} 2 + \frac{1}{4} - \frac{1}{2\pi}(2\omega_2 + 1)$$

$$\mathcal{S}_{-3} = \frac{1}{3} - \frac{10}{9\pi}$$

$$\mathcal{S}_{-4} = \frac{9}{32}\operatorname{logh} 2 + \frac{11}{128} - \frac{1}{32\pi}(18\omega_2 + 13)$$

$$\mathcal{S}_{-5} = \frac{1}{5} - \frac{178}{225\pi}$$

$$\mathcal{S}_{-6} = \frac{25}{128}\operatorname{logh} 2 + \frac{71}{1536} - \frac{1}{128\pi}(50\omega_2 + 43)$$

Series No.

(386) $A_n = \dfrac{1\cdot 3\cdot 5\ldots 2n-1}{2\cdot 4\cdot 6\ldots 2n}$

(387) $\sum_{1}^{\infty} A_n \dfrac{1}{4n^2-1}$

(388) $\sum_{1}^{\infty} A_n \dfrac{1}{2n+1}$

(389) $\sum_{1}^{\infty} (-1)^n A_n \dfrac{1}{2n+1}$

(390) $\sum_{1}^{\infty} A_n{}^2 \dfrac{4n+1}{(2n-1)(2n+2)}$

(391) $\sum_{1}^{\infty} (-1)^{n+1} A_n{}^3 \dfrac{4n+1}{(2n-1)(2n+2)}$

(392) $\sum_{1}^{\infty} (-1)^n A_n{}^3 (4n+1)$

(393) $\sum_{1}^{\infty} A_n{}^4 \dfrac{4n+1}{(2n-1)(2n+2)}$

(394) $\sum_{n=1}^{\infty} \dfrac{1}{m^2-n^2}$

(395) $\sum_{1}^{\infty} \dfrac{(-1)^{n-1}}{m^2-n^2}$

(396) $\sum_{2}^{\infty} \dfrac{n-1}{n!}$

(397) $\sum_{1}^{\infty} \dfrac{1}{4n^2-1}$

(398) $\sum_{1}^{\infty} \dfrac{12n^2-1}{n(4n^2-1)^2}$

Reference

$$= \frac{(2n-1)!}{2^{2n-1}n!(n-1)!}$$ T. 143

$$= 1 - \frac{\pi}{4}$$

$$= \frac{\pi}{2} - 1$$

$$= \operatorname{logh}(1 + \sqrt{2}) - 1$$

$$= \frac{1}{2}$$

$$= \frac{2}{\pi} - \frac{1}{2}$$

$$= \frac{2}{\pi} - 1$$

$$= \frac{1}{2} - \frac{4}{\pi^2}$$

$$= -\frac{3}{4m^2}$$ where m is an integer and $n = m$ omitted A. 67

$$= \frac{3}{4m^2}$$ where n is even and $n = m$ omitted A. 67

$$= 1$$ T. 143

$$= \frac{1}{2}$$ T. 141

$$= 2 \operatorname{logh} 2$$ T. 143

1 32
4 128
3 128 = 1/2 Slem
3 64

Series No.

(399) $1 - \frac{1}{2^2} + \frac{1}{2^4} - \frac{1}{2^6} - \ldots \infty$

(400) $\sum_{2}^{\infty} \frac{1}{(n^2 - 1)^2} = \left[\frac{1}{(1\cdot 3)^2} + \frac{1}{(2\cdot 4)^2} + \frac{1}{(3\cdot 5)^2} + \ldots \infty\right]$

(401) $\sum_{2}^{\infty} \frac{(-1)^n}{(n^2 - 1)^2}$

(402)† $x + aB_2(x) + \frac{a^2}{2!} B_3(x) + \frac{a^3}{3!} B_4(x) + \ldots \infty$

(403)† $1 + a\left(x - \frac{1}{2}\right) + a^2 A_2(x) + \frac{a^3}{2!} A_3(x) + \ldots \infty$

(404) $\frac{1}{n+1} + \frac{1}{2} + \frac{n}{2!} B_1{}^* - \frac{n(n-1)(n-2)}{4!} B_3{}^* + \ldots \infty$

This series may be used to evaluate $B_n{}^*$ by putting $n = 2, 4, 6$, etc.

(405) $\frac{1}{2}\left(\frac{2x}{1 + x^2}\right) + \frac{1}{2\cdot 4}\left(\frac{2x}{1 + x^2}\right)^3 + \frac{1\cdot 3}{2\cdot 4\cdot 6}\left(\frac{2x}{1 + x^2}\right)^5 + \ldots \infty$

(406) $\sum_{1}^{\infty} \mathcal{S}_n \frac{x^n}{n!}$

(407) $\sum_{1}^{\infty} (-1)^{n-1} \mathcal{S}_n \frac{2^n}{n!}$

(408) $1 + nx + \frac{n^2x^2}{2!} + \frac{n(n^2 - 1^2)}{3!} x^3 + \frac{n^2(n^2 - 2^2)}{4!} x^4 + \frac{n(n^2 - 1^2)(n^2 - 3^2)}{5!} x^5 + \ldots \infty$

(409) $1 + \frac{1}{2^2}\left(\frac{1}{2}\right)^2 + \frac{1}{3^2}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 + \ldots \infty$

(410) $1 + \frac{1}{3}\cdot 1^2 + \frac{1}{5}\left(\frac{2}{3}\right)^2 + \frac{1}{7}\left(\frac{2\cdot 4}{3\cdot 5}\right)^2 + \ldots \infty$

† For values of $A_n(x)$ and $B_n(x)$, see No. (1146).

Reference

$= \frac{4}{5}$ T. 144

$= \frac{\pi^2}{12} - \frac{11}{16}$

$= \frac{\pi^2}{24} - \frac{5}{16}$

$= \frac{\epsilon^{ax} - 1}{\epsilon^a - 1}$ where x is a positive integer AE. 6

$= \frac{a\epsilon^{ax}}{\epsilon^a - 1}$ AE. 20

$= 1$ Y. 109

$= x$ where $1 > x > -1$ Y. 459

$= \frac{1}{x}$ where $x > 1$

$= \frac{1}{4} x\epsilon^x(x^3 + 8x^2 + 14x + 4)$ where $S_n = 1^3 + 2^3 + 3^3 + \dots n^3$
A. 197

$= 0$ where $S_n = 1^3 + 2^3 + 3^3 + \dots n^3$ A. 197

$= (x + \sqrt{1 + x^2})^n$ Y. 107

$= \frac{11}{\pi} - 4$ 1Z. 360

$= \frac{\pi}{2}$ 1Z. 360

Series No.

(411) $\frac{1}{3^2}\cdot 1^2 + \frac{1}{5^2}\left(\frac{2}{3}\right)^2 + \frac{1}{7^2}\left(\frac{2\cdot 4}{3\cdot 5}\right)^2 + \ldots \infty$

(412) $$\frac{x(a+x)}{a} - \frac{x^2(a+x)^2}{a^3} + \frac{4}{1\cdot 2}\frac{x^3(a+x)^3}{a^5} + \ldots + (-1)^{n-1}\frac{(2n-2)!}{n!(n-1)!}\frac{x^n(a+x)^n}{a^{2n-1}} + \ldots \infty$$

(413) $$1 - ax(1-x) + \frac{a(a-3)}{2!}x^2(1-x)^2 - \frac{a(a-4)(a-5)}{3!}x^3(1-x)^3 + \ldots$$

(414) $$\operatorname*{Lim}_{x\to 1} \sum_{1}^{\infty} (-1)^{n-1}\frac{x^n}{n(1+x^n)}$$

(415) $$\operatorname*{Lim}_{x\to 1} \sum_{1}^{\infty} \frac{x^n}{1-x^n}$$

(416) $$\operatorname*{Lim}_{x\to 1} \left(\frac{x}{1-x} - \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} + \ldots \infty\right)$$

XIV. Trigonometrical Summations

(417) $\sin\theta + \sin 2\theta + \sin 3\theta +$ to n terms

(418) $\cos\theta + \cos 2\theta + \cos 3\theta +$ to n terms

(419) $\sin\theta + \sin 3\theta + \sin 5\theta + \ldots + \sin(2n-1)\theta$

(420) $\cos\theta + \cos 3\theta + \cos 5\theta + \ldots + \cos(2n-1)\theta$

(421) $\cos\theta + \sin 3\theta + \cos 5\theta + \ldots + \sin(4n-1)\theta$

Reference

$= \pi - 3$ 1Z. 360

$= x$ A. 199

$= (1 - x)^a$ where $|x(1 - x)| < \frac{1}{4}$ A. 199

$= \frac{1}{2}$ logh 2 A. 201

$\approx \frac{1}{1 - x}$ logh $\frac{1}{1 - x}$ A. 201

$\approx \frac{\pi}{4} \cdot \frac{1}{1 - x}$ A. 201

$= \sin \frac{1}{2}(n + 1)\theta \sin \frac{n\theta}{2} \operatorname{cosec} \frac{\theta}{2}$ E. 283

$= \cos \frac{1}{2}(n + 1)\theta \sin \frac{n\theta}{2} \operatorname{cosec} \frac{\theta}{2}$ E. 283

$= \sin^2 n\theta \cdot \operatorname{cosec} \theta$ E. 283

$= \frac{1}{2} \sin 2n\theta \cdot \operatorname{cosec} \theta$ E. 287

$= \sin 2n\theta \{\cos 2n\theta + \sin 2n\theta\} \times \{\cos \theta + \sin \theta\} \operatorname{cosec} 2\theta$ E. 288

Series No.

(422) $\operatorname{cosec}\theta + \operatorname{cosec}2\theta + \operatorname{cosec}4\theta + \ldots n$ terms

(423) $\cos\frac{\theta}{2} + \cos 2\theta + \cos\frac{7\theta}{2} + \ldots n$ terms

(424) $\cos\frac{\pi}{2n+1} + \cos\frac{3\pi}{2n+1} + \ldots n$ terms

(425) $1 - 2\cos\theta + 3\cos 2\theta - 4\cos 3\theta + \ldots n$ terms

(426) $3\sin\theta + 5\sin 2\theta + 7\sin 3\theta + \ldots n$ terms

(427) $\sum_{1}^{n-1} k\sin k\theta$

(428) $\sum_{1}^{n-1} k\cos k\theta$

(429) $\sum_{1}^{n} (-1)^k \cos k\theta$

(430) $\sum_{1}^{n} \sin(2k-1)\theta$

(431) $\sum_{1}^{n+1} (-1)^{k-1}\sin(2k-1)\theta$

(432) $\tan\theta + \tan\left(\theta + \frac{\pi}{n}\right) + \tan\left(\theta + \frac{2\pi}{n}\right) + \ldots n$ terms

Reference

$$= \cot\frac{\theta}{2} - \cot 2^{n-1}\theta$$ E. 125

$$= \cos\frac{1}{4}(3n-1)\theta \sin\frac{3n\theta}{4} \operatorname{cosec}\frac{3\theta}{4}$$ E. 287

$$= \frac{1}{2}$$ E. 288

$$= \frac{\cos\theta + (-1)^{n-1}\{(n+1)\cos(n-1)\theta + n\cos n\theta\}}{2(1+\cos\theta)}$$ E. 117

$$= \frac{\sin\theta + (2n+3)\sin n\theta - (2n+1)\sin(n+1)\theta}{2(1-\cos\theta)}$$ E. 117

$$= \frac{\sin n\theta}{4\sin^2\frac{\theta}{2}} - \frac{n\cos\left(\frac{2n-1}{2}\right)\theta}{2\sin\frac{\theta}{2}}$$ T. 82

$$= \frac{n\sin\left(\frac{2n-1}{2}\right)\theta}{2\sin\frac{\theta}{2}} - \frac{1-\cos n\theta}{4\sin^2\frac{\theta}{2}}$$ T. 82

$$= -\frac{1}{2} + (-1)^n \frac{\cos\left(\frac{2n+1}{2}\right)\theta}{2\cos\frac{\theta}{2}}$$ T. 82

$$= \frac{\sin^2 n\theta}{\sin\theta}$$ T. 82

$$= (-1)^n \frac{\sin(2n+2)\theta}{2\cos\theta}$$ T. 82

$$= -n\cot\left(\frac{n\pi}{2} + n\theta\right)$$ E. 73

Series No.

(433) $\cot\theta + \cot\left(\theta + \frac{\pi}{n}\right) + \cot\left(\theta + \frac{2\pi}{n}\right) + \ldots n$ terms

(434) $\sum_{0}^{n-1} \cos\frac{2\pi k^2}{n}$

(435) $\sum_{1}^{n-1} \sin\frac{2\pi k^2}{n}$

(436) $\sum_{1}^{n-1} \sin\frac{\pi k}{n}$

(437) $\sum_{1}^{n} \sin^2 k\theta$

(438) $\sum_{0}^{n} \cos^2 k\theta$

(439) $\frac{1}{\sin^2\theta} + \frac{1}{\sin^2 2\theta} + \ldots + \frac{1}{\sin^2\frac{n-1}{2}\theta}$

(440) $\frac{1}{\sin^2\theta} + \frac{1}{\sin^2 2\theta} + \ldots + \frac{1}{\sin^2\frac{n-2}{2}\theta}$

(441) $\frac{1}{\sin^2\theta} + \frac{1}{\sin^2 3\theta} + \ldots + \frac{1}{\sin^2(n-2)\theta}$

(442) $\frac{1}{\sin^2\theta} + \frac{1}{\sin^2 3\theta} + \ldots + \frac{1}{\sin^2(n-1)\theta}$

(443) $\sum_{1}^{n-1} \operatorname{cosec}^2\left(\frac{r\pi}{n}\right)$

(444) $\operatorname{cosec}^2\theta + \operatorname{cosec}^2\left(\theta + \frac{2\pi}{n}\right) + \ldots n$ terms

(445) $\tan^2\theta + \tan^2\left(\theta + \frac{\pi}{n}\right) + \ldots n$ terms

Reference

$= n \cot n\theta$ E. 73

$= \frac{\sqrt{n}}{2}\left(1 + \cos\frac{n\pi}{2} + \sin\frac{n\pi}{2}\right)$ T. 83

$= \frac{\sqrt{n}}{2}\left(1 + \cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right)$ T. 83

$= \cot\frac{\pi}{2n}$ T. 83

$= \frac{n}{2} - \frac{\cos(n+1)\theta \sin n\theta}{2\sin\theta}$ T. 82

$= \frac{n+2}{2} + \frac{\cos(n+1)\theta \sin n\theta}{2\sin\theta}$ T. 82

$= \frac{n^2-1}{6}$ where n is odd A. 210

$= \frac{n^2-4}{6}$ where n is even A. 210

$= \frac{n^2-1}{2}$ where n is odd A. 211

$= \frac{n^2}{2}$ where n is even A. 211

$= \frac{1}{3}(n^2 - 1)$ where n is odd A. 223

$= n^2 \operatorname{cosec}^2 n\theta$ where n is odd

$= \frac{1}{2} n^2 \operatorname{cosec}^2 \frac{n\theta}{2}$ where n is even E. 73

$= n^2 \cot^2\left(\frac{n\pi}{2} + n\theta\right) + n(n-1)$ E. 73

Series No.

(446) $\sin^3 \theta + \sin^3 2\theta + \sin^3 3\theta + \ldots n$ terms

(447) $\cos^3 \theta + \cos^3 2\theta + \cos^3 3\theta + \ldots n$ terms

(448) $\sin^4 \theta + \sin^4 2\theta + \sin^4 3\theta + \ldots n$ terms

(449) $\cos^4 \theta + \cos^4 2\theta + \cos^4 3\theta + \ldots n$ terms

(450) $\sum_{1}^{\frac{n-1}{2}} \tan^4 \left(\frac{r\pi}{n}\right)$

(451) $\cot^2 \frac{2\pi}{2n} + \cot^2 \frac{4\pi}{2n} + \ldots + \cot^2 \frac{(n-1)\pi}{2n}$

(452) $\cot^4 \frac{2\pi}{2n} + \cot^4 \frac{4\pi}{2n} + \ldots + \cot^4 \frac{(n-1)\pi}{2n}$

(453) $1 + a \cos \theta + a^2 \cos 2\theta + \ldots + a^{n-1} \cos (n-1)\theta$

(454) $\sum_{0}^{n-1} a^k \cos k\theta$

Nos. (453) and (454) are equal.

(455) $\sum_{1}^{n-1} a^k \sin k\theta$

Reference

$$= \frac{3}{4}\sin\frac{1}{2}(n+1)\theta\sin\frac{n\theta}{2}\operatorname{cosec}\frac{\theta}{2} - \frac{1}{4}\sin\frac{3}{2}(n+1)\theta\sin\frac{3n\theta}{2}\operatorname{cosec}\frac{3\theta}{2}$$

E. 288

$$= \frac{3}{4}\cos\frac{1}{2}(n+1)\theta\sin\frac{n\theta}{2}\operatorname{cosec}\frac{\theta}{2} + \frac{1}{4}\cos\frac{3}{2}(n+1)\theta\sin\frac{3n\theta}{2}\operatorname{cosec}\frac{3\theta}{2}$$

E. 285

$$= \frac{1}{8}[3n - 4\cos(n+1)\theta\sin n\theta\operatorname{cosec}\theta + \cos 2(n+1)\theta\sin 2n\theta\operatorname{cosec}2\theta]$$

E. 288

$$= \frac{1}{8}[3n + 4\cos(n+1)\theta\sin n\theta\operatorname{cosec}\theta + \cos 2(n+1)\theta\sin 2n\theta\operatorname{cosec}2\theta]$$

E. 288

$$= \frac{1}{6}n(n-1)(n^2+n-3) \quad \text{where } n \text{ is odd}$$

A. 223

$$= \frac{1}{6}(n-1)(n-2) \quad \text{where } n \text{ is odd}$$

O. 349

$$= \frac{1}{90}(n-1)(n-2)(n^2+3n-13) \quad \text{where } n \text{ is odd}$$

O. 349

$$= \frac{1 - a\cos\theta + a^{n+1}\cos(n-1)\theta - a^n\cos n\theta}{1 - 2a\cos\theta + a^2}$$

$$= \frac{(1 - a\cos\theta)(1 - a^n\cos n\theta) + a^{n+1}\sin\theta\sin n\theta}{1 - 2a\cos\theta + a^2}$$

T. 82

$$= \frac{a\sin\theta(1 - a^n\cos n\theta) - (1 - a\cos\theta)a^n\sin n\theta}{1 - 2a\cos\theta + a^2}$$

T. 82

Series No.

(456) $a \sin\theta + 2a^2 \sin 2\theta + 3a^3 \sin 3\theta + \ldots$ n terms

(457) $\cos^3\theta - \frac{1}{3}\cos^3 3\theta + \frac{1}{3^2}\cos^3 3^2\theta + \frac{1}{3^3}\cos^3 3^3\theta + \ldots$ to n terms

(458) $\cos\frac{\pi}{n} + \cos\frac{2\pi}{n} + \ldots + \cos\frac{(2n-1)\pi}{n}$

(459) $\sum_{0}^{n} \frac{1}{2^n}\tan\frac{\theta}{2^n}$

(460) $\sum_{1}^{n} \left(\frac{1}{2^n}\sec\frac{\theta}{2^n}\right)^2$

(461) $\sum_{1}^{n} \left(2^n \sin^2\frac{\theta}{2^n}\right)^2$

(462) $\sum_{0}^{n} \left(\frac{1}{2^{2n}}\tan^2\frac{\theta}{2^n}\right)$

(463) $\sum_{1}^{\frac{n-1}{2}} \frac{1}{n^2\sin^2\left(\frac{r\pi}{n}\right)}$

(464) $\sum_{1}^{n-1} \frac{1}{n^2\sin^2\left(\frac{r\pi}{2n}\right)}$

(465) $\sin\theta + \sin(\theta+\beta) + \sin(\theta+2\beta) + \ldots$ n terms

(466) $\cos\theta + \cos(\theta+\beta) + \cos(\theta+2\beta) + \ldots$ n terms

(467) $\sin\theta - \sin(\theta+\beta) + \sin(\theta+2\beta) - \ldots$ n terms

(468) $\cos\theta - \cos(\theta+\beta) + \cos(\theta+2\beta) - \ldots$ $2n$ terms

Reference

$$= [a \sin \theta - a^3 \sin \theta - (n+1)a^{n+1} \sin (n+1)\theta + 2(n+1)a^{n+2} \sin n\theta - (n+1)a^{n+3} \sin (n-1)\theta + na^{n+2} \sin (n+2)\theta - 2na^{n+3} \sin (n+1)\theta + na^{n+4} \sin n\theta]/(1 - 2a \cos \theta + a^2)^2$$ D. 502

$$= \frac{1}{4}\left\{3 \cos \theta + \left(-\frac{1}{3}\right)^{n-1} \cos 3^n\theta\right\}$$ E. 126

$$= -1$$

$$= \frac{1}{2^n} \cot \frac{\theta}{2^n} - 2 \cot 2\theta$$ T. 83

$$= \operatorname{cosec}^2 \theta - \left(\frac{1}{2^n} \operatorname{cosec} \frac{\theta}{2^n}\right)^2$$ T. 82

$$= \left(2^n \sin \frac{\theta}{2^n}\right)^2 - \sin^2 \theta$$ T. 82

$$= \frac{2^{2n+2} - 1}{3 \cdot 2^{2n-1}} + 4 \cot^2 2\theta - \frac{1}{2^{2n}} \cot \frac{\theta}{2^n}$$ T. 83

$$= \frac{n^2 - 1}{6n^2} \quad \text{where } n \text{ is odd}$$ A. 218

$$= \frac{1}{2} \quad \text{where } r \text{ is odd and } n \text{ is even}$$ A. 218

$$= \sin\left\{\theta + \frac{1}{2}(n-1)\beta\right\} \sin \frac{n\beta}{2} \operatorname{cosec} \frac{\beta}{2}$$ E. 282

$$= \cos\left\{\theta + \frac{1}{2}(n-1)\beta\right\} \sin \frac{n\beta}{2} \operatorname{cosec} \frac{\beta}{2}$$ E. 283

$$= \sin\left\{\theta + \frac{n-1}{2}(\beta + \pi)\right\} \sin \frac{n(\beta + \pi)}{2} \sec \frac{\beta}{2}$$ E. 285

$$= \sin\left\{\theta + \left(n - \frac{1}{2}\right)\beta\right\} \sin n\beta \sec \frac{\beta}{2}$$ E. 288

Series No.

(469) $\sin\theta\cdot\sin 2\theta + \sin 2\theta\cdot\sin 3\theta + \ldots$ n terms

(470) $\cos\theta\cdot\sin 2\theta + \sin 2\theta\cdot\cos 3\theta + \ldots$ $2n$ terms

(471) $\sin\theta\cdot\sin 3\theta + \sin 2\theta\cdot\sin 4\theta + \ldots$ n terms

(472) $\cos\theta\sin\beta + \cos 3\theta\sin 2\beta + \cos 5\theta\sin 3\beta + \ldots$ n terms

(473) $\sum_{1}^{n} r\sin(\phi + r\theta)$

(474) $-6\sin(a + b) - 5\sin(2a + b) - \ldots + (n - 7)\sin(na + b)$

(475)† $\sin\theta + a\sin(\theta + \beta) + a^2\sin(\theta + 2\beta) + \ldots$ n terms

(476) $\tan\theta\tan(\theta + \beta) + \tan(\theta + \beta)\tan(\theta + 2\beta) + \ldots$ n terms

(477) $\operatorname{cosec}\theta\operatorname{cosec}2\theta + \operatorname{cosec}2\theta\operatorname{cosec}3\theta + \ldots$ n terms

(478) $\sec\theta\sec 2\theta + \sec 2\theta\sec 3\theta + \ldots$ n terms

(479) $\sum_{0}^{n}\sin(\theta + k\beta)$

† Sum to infinity $= \dfrac{\sin\theta - a\sin(\theta - \beta)}{1 - 2a\cos\beta + a^2}$ where $a < 1$.

Reference

$$= \frac{1}{4}\{(n+1)\sin 2\theta - \sin 2(n+1)\theta\}\operatorname{cosec}\theta$$ E. 288

$$= \frac{1}{2}\sin 2(n+1)\theta\cdot\sin 2n\theta\operatorname{cosec}\theta$$ E. 288

$$= \frac{n}{2}\cos 2\theta - \frac{1}{2}\cos(n+3)\theta\cdot\sin n\theta\cdot\operatorname{cosec}\theta$$ E. 288

$$= \frac{1}{2}\sin\left\{n\theta + \frac{1}{2}(n+1)\beta\right\}\sin\frac{n}{2}(2\theta+\beta)\operatorname{cosec}\frac{1}{2}(2\theta+\beta) - \frac{1}{2}\sin\left\{n\theta - \frac{1}{2}(n+1)\beta\right\}\sin\frac{n}{2}(2\theta-\beta)\operatorname{cosec}\frac{1}{2}(2\theta-\beta)$$ E. 286

$$= \frac{(n+1)\sin(\phi+n\theta) - \sin\phi - n\sin(\phi+\overline{n+1}\theta)}{2(1-\cos\theta)}$$

$$= (n-7)\sin(na+b) + (n-8)\frac{\sin\left(na+b-\dfrac{a+\pi}{2}\right)}{2\sin a/2} - \frac{\sin\{na+b-(a+\pi)\}}{2(\sin a/2)^2} - \frac{7\sin(a+b) - 6\sin b}{(2\sin a/2)^2}$$

$$= \frac{\sin\theta - a\sin(\theta-\beta) - a^n\sin(\theta+n\beta) + a^{n+1}\sin\{\theta+(n-1)\beta\}}{1-2a\cos\beta+a^2}$$ E. 117

$$= \frac{\tan(\theta+n\beta) - \tan\theta - n\tan\beta}{\tan\beta}$$ E. 124

$$= \operatorname{cosec}\theta\{\cot\theta - \cot(n+1)\theta\}$$ E. 125

$$= \operatorname{cosec}\theta\{\tan(n+1)\theta - \tan\theta\}$$ E. 125

$$= \sin\left(\theta+\frac{n\beta}{2}\right)\sin\left(\frac{n+1}{2}\right)\beta\operatorname{cosec}\frac{\beta}{2}$$ T. 82

Series No.

(480) $\sum_{0}^{n} \cos(\theta + k\beta)$

(481) $1 - \frac{n^2 - 1^2}{2!}\sin^2\theta + \frac{(n^2 - 1^2)(n^2 - 3^2)}{4!}\sin^4\theta + \ldots$
$+ (-1)^{(n-1)/2}2^{n-1}\sin^{n-1}\theta$

(482) $n\sin\theta - \frac{n(n^2 - 1^2)}{3!}\sin^3\theta + \frac{n(n^2 - 1^2)(n^2 - 3^2)}{5!}\sin^5\theta + \ldots$
$+ (-1)^{(n-1)/2}2^{n-1}\sin^n\theta$

(482a) $1 - \frac{n^2 - 1^2}{2!}\cos^2\theta + \frac{(n^2 - 1^2)(n^2 - 3^2)}{4!}\cos^4\theta$
$- \frac{(n^2 - 1^2)(n^2 - 3^2)(n^2 - 5^2)}{6!}\cos^6\theta + \ldots$
$+ (-1)^{(n-1)/2}(2\cos\theta)^{n-1}$

(482b) $n\cos\theta - \frac{n(n^2 - 2^2)}{3!}\cos^3\theta$
$+ \frac{n(n^2 - 2^2)(n^2 - 4^2)}{5!}\cos^5\theta - \ldots$
$+ (-1)^{n/2+1}(2\cos\theta)^{n-1}$

(482c) $n\cos\theta - \frac{n(n^2 - 1^2)}{3!}\cos^3\theta$
$+ \frac{n(n^2 - 1^2)(n^2 - 3^2)}{5!}\cos^5\theta - \ldots$
$+ (-1)^{(n-1)/2}2^{n-1}\cos^n\theta$

(482d) $1 - \frac{n^2}{2!}\cos^2\theta + \frac{n^2(n^2 - 2^2)}{4!}\cos^4\theta$
$- \frac{n^2(n^2 - 2^2)(n^2 - 4^2)}{6!}\cos^6\theta + \ldots + (-1)^{n/2}2^{n-1}\cos^n\theta$

Reference

$$= \cos\left(\theta + \frac{n\beta}{2}\right) \sin\left(\frac{n+1}{2}\right) \beta \operatorname{cosec} \frac{\beta}{2}$$ T. 82

$$= \frac{\cos n\theta}{\cos n} \quad \text{where } n \text{ is odd}$$ A. 204

$$= \sin n\theta \quad \text{where } n \text{ is odd}$$ A. 205

$$= (-1)^{(n-1)/2} \frac{\sin n\theta}{\sin \theta} \quad \text{where } n \text{ is odd}$$ E. 64; A. 204

$$= (-1)^{n/2+1} \frac{\sin n\theta}{\sin \theta} \quad \text{where } n \text{ is even}$$ E. 65; A. 204

$$= (-1)^{(n-1)/2} \cos n\theta \quad \text{where } n \text{ is odd}$$ E. 67; A. 204

$$= (-1)^{n/2} \cos n\theta \quad \text{where } n \text{ is even}$$ E. 68; A. 204

Series No.

(483) $n \sin\theta - \frac{n(n^2-2^2)}{3!}\sin^3\theta + \frac{n(n^2-2^2)(n^2-4^2)}{5!}\sin^5\theta + \dots + (-1)^{n/2+1}(2\sin\theta)^{n-1}$

(484)† $1 - \frac{n^2}{2!}\sin^2\theta + \frac{n^2(n^2-2^2)}{4!}\sin^4\theta + \dots + (-1)^{n/2}2^{n-1}\sin^n\theta$

(485) $\sum_{1}^{\frac{n-1}{2}} \{\cot(\theta + ra) + \cot(\theta - ra)\}$

(486) $\sum_{1}^{\frac{n-1}{2}} \{\operatorname{cosec}^2(\theta + ra) + \operatorname{cosec}^2(\theta - ra)\}$

(487) $\frac{\sin\theta}{2\cos\theta - 1} + \frac{2\sin 2\theta}{2\cos 2\theta - 1} + \frac{2^2\sin 2^2\theta}{2\cos 2^2\theta - 1} + \dots$ n terms

(488) $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} + \tan^{-1}\frac{1}{21} + \dots + \tan^{-1}\frac{1}{1 + n(n+1)}$

(489) $2\sum_{1}^{\frac{n-2}{2}} \frac{x - a\cos\frac{2r\pi}{n}}{x^2 - 2ax\cos\frac{2r\pi}{n} + a^2}$

(490) $2\sum_{1}^{\frac{n-1}{2}} \frac{x - a\cos\frac{2r\pi}{n}}{x^2 - 2ax\cos\frac{2r\pi}{n} + a^2}$

† This summation can be carried to infinity, the sum being $\cos\frac{1}{2}n\pi$, where the value of n is unrestricted, and $\theta = \frac{\pi}{2}$.

Reference

$= \dfrac{\sin n\theta}{\cos\theta}$ where n is even — A. 205

$= \cos n\theta$ where n is even — A. 204

$= n \cot n\theta - \cot\theta$ where $na = \pi$ — A. 217

$= n^2 \operatorname{cosec}^2 n\theta - \operatorname{cosec}^2\theta$ where $na = \pi$ — A. 217

$= \dfrac{2^n \sin 2^n\theta}{2^n \cos 2^n\theta + 1} - \dfrac{\sin\theta}{2\cos\theta + 1}$ — D. 330

$= \tan^{-1} \dfrac{n}{n+2}$ — E. 126

$= \dfrac{nx^{n-1}}{x^n - a^n} - \dfrac{1}{x-a} - \dfrac{1}{x+a}$ where n is even — Y. 55

$= \dfrac{nx^{n-1}}{x^n - a^n} - \dfrac{1}{x-a}$ where n is odd — Y. 55

A. 207

Series No.

(491) $\lim\limits_{n=\infty}\left[\frac{1}{n}\left\{\sec\frac{\theta}{n}+\sec\frac{2\theta}{n}+\ldots+\sec\frac{(n-1)\theta}{n}\right\}\right]$

(492) $\sum\limits_{1}^{n}\tan^{-1}\left(\sec\frac{2r\pi}{2n+1}\sinh\theta\right)$

(493) $(2\cos\theta)^{n-1}-(n-2)(2\cos\theta)^{n-3}$
$+\frac{(n-3)(n-4)}{2!}(2\cos\theta)^{n-5}+$
$\ldots+(-1)^{(n-1)/2}$
$\ldots+(-1)^{n/2-1}(n\cos\theta)$

(494) $(2\cos\theta)^{n}-n(2\cos\theta)^{n-2}+\frac{n(n-3)}{2!}(2\cos\theta)^{n-4}+$
$\ldots+(-1)^{(n-1)/2}2n\cos\theta$ (n is odd)
$\ldots+2(-1)^{n/2}$ (n is even)

(495) $\frac{1}{\cos\theta+\cos 3\theta}+\frac{1}{\cos 3\theta+\cos 5\theta}+\ldots$ n terms

(496) $\lim\limits_{n=\infty}\left[\frac{1}{n}\left\{\sin^{2k}\frac{\pi}{2n}+\sin^{2k}\frac{2\pi}{2n}+\ldots+\sin^{2k}\frac{\pi}{2}\right\}\right]$

(497) $\lim\limits_{x\to 1}[1-x\cos\theta+x^{4}\cos 2\theta-x^{9}\cos 3\theta+\ldots\infty]$

(498) $\lim\limits_{x\to 1}[x\sin\theta-x^{4}\sin 2\theta+x^{9}\sin 3\theta-\ldots\infty]$

(499) $\sum\limits_{1}^{\infty}a^{n}\sin n\theta$

(500) $\sum\limits_{0}^{\infty}a^{n}\cos n\theta$

(501) $\cos\theta+a\cos 3\theta+a^{2}\cos 5\theta+\ldots a^{n}\cos(2n+1)\theta+\ldots\infty$

Reference

$$= \frac{1}{\theta} \operatorname{logh} \tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) \quad \text{where } \pi > \theta > \frac{\pi}{2}$$ 1Z. 355

$$= \tan^{-1} \frac{\sinh n\theta}{\cosh (n+1)\theta} \quad \text{where } n \text{ is even}$$ A. 528

$$= \sin n\theta \cdot \operatorname{cosec} \theta \quad \text{where } n \text{ is odd}$$
$$= \sin n\theta \cdot \operatorname{cosec} \theta \quad \text{where } n \text{ is even}$$ E. 61

$$= 2 \cos n\theta$$ E. 63

$$= \frac{1}{2} \operatorname{cosec} \theta \{\tan (n+1)\theta - \tan \theta\}$$ E. 125

$$= \frac{(2k-1)(2k-3)\ldots 1}{2k(2k-2)\ldots 2} \quad \text{where } k \text{ is a positive integer}$$ 1Z. 326

$$= \frac{1}{2}$$ A. 276

$$= \frac{1}{2} \tan \frac{\theta}{2}$$ A. 276

$$= \frac{a \sin \theta}{1 - 2a \cos \theta + a^2} \quad \text{where } a^2 < 1$$ T. 139

$$= \frac{1 - a \cos \theta}{1 - 2a \cos \theta + a^2} \quad \text{where } a^2 < 1$$ T. 139

$$= \frac{(1-a) \cos \theta}{1 - 2a \cos 2\theta + a^2} \quad \text{where } |a| < 1$$ A. 223

Series No.

(502) $\tan\theta + \tan\left(\theta + \frac{2\pi}{5}\right) + \tan\left(\theta + \frac{4\pi}{5}\right) + \dots \tan\left(\theta + \frac{8\pi}{5}\right)$

(503) $\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{3}\cos 3\theta + \dots \infty$

(504) $\cos\theta - \frac{1}{2}\cos 2\theta + \frac{1}{3}\cos 3\theta + \dots \infty$

(505) $\cos\theta + \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta + \dots \infty$

(506) $\cos\theta - \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta + \dots \infty$

(507) $\cos 2\theta + \frac{1}{2}\cos 4\theta + \frac{1}{3}\cos 6\theta + \dots \infty$

(508) $\sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta + \dots \infty$

(509) $\sin\theta - \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta + \dots \infty$

(510) $\sin\theta + \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta + \dots \infty$

(511) $\sin\theta - \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta - \dots \infty$

(512) $\sin 2\theta + \frac{1}{2}\sin 4\theta + \frac{1}{3}\sin 6\theta + \dots \infty$

	Reference
$= 5 \tan 5\theta$	E. 191
$= -\operatorname{logh} 2 \sin \frac{\theta}{2}$ where $0 < \theta < 2\pi$	A. 356
$= \operatorname{logh} 2 \cos \frac{\theta}{2}$ where $-\pi < \theta < \pi$	A. 356
$= \frac{1}{2} \operatorname{logh} \cot \frac{\theta}{2}$ where $0 < \theta < \pi$	A. 356
$= \frac{1}{2} \operatorname{logh} \left(-\cot \frac{\theta}{2}\right)$ where $\pi < \theta < 2\pi$	A. 356
$= \frac{\pi}{4}$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	A. 359
$= -\operatorname{logh} (2 \sin \theta)$ where $0 < \theta < \pi$	A. 356
$= \frac{1}{2} (\pi - \theta)$ where $0 < \theta < 2\pi$	A. 356
$= \frac{\theta}{2}$ where $-\pi < \theta < \pi$	A. 356
$= \frac{\pi}{4}$ where $0 < \theta < \pi$	A. 356
$= -\frac{\pi}{4}$ where $\pi < \theta < 2\pi$	A. 356
$= \frac{1}{2} \operatorname{logh} (\sec \theta + \tan \theta)$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	A. 359
$= \frac{1}{2} (\pi - 2\theta)$ where $0 < \theta < \pi$	A. 356

Series No.

(513) $\cos\frac{\pi}{3} + \frac{1}{3}\cos\frac{2\pi}{3} + \frac{1}{5}\cos\frac{3\pi}{3} + \frac{1}{7}\cos\frac{4\pi}{3} + \ldots \infty$

(514) $\tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \ldots \infty$

(515) $\frac{1}{2}\sin\theta - \frac{2}{5}\sin 2\theta + \frac{3}{10}\sin 3\theta - \frac{4}{17}\sin 4\theta + \ldots \infty$

(516) $\cos\theta + \frac{1}{2^2}\cos 2\theta + \frac{1}{3^2}\cos 3\theta + \ldots \infty$

(517) $\cos\theta - \frac{1}{2^2}\cos 2\theta + \frac{1}{3^2}\cos 3\theta - \ldots \infty$

(518) $\cos\theta + \frac{1}{3^2}\cos 3\theta + \frac{1}{5^2}\cos 5\theta + \ldots \infty$

(519) $\sin\theta - \frac{1}{3^2}\sin 3\theta + \frac{1}{5^2}\sin 5\theta - \ldots \infty$

(520) $\sum_{1}^{\infty} \frac{\sin^2 n\theta}{n^2}$

(521) $\sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{2^2}\sin 3\theta + \ldots \infty$

(522) $\sin\frac{\theta}{2} - \frac{1}{3^2}\sin\frac{3}{2}\theta + \frac{1}{5^2}\sin\frac{5}{2}\theta + \ldots \infty$

(523) $\sin^2\theta - \frac{1}{2}\sin^2 2\theta + \frac{1}{3}\sin^2 3\theta - \ldots \infty$

(524) $\sin 2\theta + \frac{1}{2}\sin 3\theta + \frac{1}{3}\sin 4\theta + \ldots \infty$

	Reference
$= \frac{1}{8}\{2\sqrt{3}\ \text{logh}\ (2+\sqrt{3}) - \pi\}$	E. 123
$= \theta \quad \text{where } \frac{\pi}{4} > \theta > -\frac{\pi}{4}$	E. 107
$= \frac{\pi \sinh \theta}{2 \sinh \pi}$	U. 42
$= \frac{1}{4}(\theta - \pi)^2 - \frac{1}{12}\pi^2 \quad \text{where } 0 \leqslant \theta \leqslant 2\pi$	A. 360
$= \frac{\pi^2}{12} - \frac{\theta^2}{4} \quad \text{where } -\pi \leqslant \theta \leqslant \pi$	A. 360
$= \frac{\pi}{8}(\pi - 2\theta) \quad \text{where } 0 \leqslant \theta \leqslant \pi$	A. 360
$= \frac{\pi}{8}(\pi + 2\theta) \quad \text{where } -\pi \leqslant \theta \leqslant 0$	A. 360
$= \frac{1}{4}\pi\theta \quad \text{where } -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	A. 360
$= \frac{\pi}{4}(\pi - \theta) \quad \text{where } \frac{\pi}{2} \leqslant \theta \leqslant \frac{3\pi}{2}$	A. 360
$= \frac{1}{2}\theta(\pi - \theta) \quad \text{where } 0 \leqslant \theta \leqslant \pi$	Q. 163
$= \frac{1}{2}\{\pi\lvert\theta\rvert - \theta^2\} \quad \text{where } -\pi \leqslant \theta \leqslant \pi$	Q. 163
$= \frac{4 \sin \theta}{5 - 4 \cos \theta}$	E. 117
$= \frac{\pi\theta}{8}$	J. 311
$= \frac{1}{2}\ \text{logh}\ \sec \theta$	C. 334
$= -\sin\theta\ \text{logh}\left(4 \sin^2 \frac{\theta}{2}\right) + \frac{1}{2}(\pi - \theta)\cos\theta$	

Series No.

(525) $\sin\theta + \frac{1}{2^3}\sin 2\theta + \frac{1}{3^3}\sin 3\theta + \ldots \infty$

(526) $\sin\theta - \frac{1}{2^3}\sin 2\theta + \frac{1}{3^3}\sin 3\theta - \ldots \infty$

(527) $\sin\theta + \frac{1}{3^3}\sin 3\theta + \frac{1}{5^3}\sin 5\theta + \ldots \infty$

(528) $\cos\theta - \frac{1}{3^3}\cos 3\theta + \frac{1}{5^3}\cos 5\theta - \ldots \infty$

(529) $\cos\theta + \frac{1}{2^4}\cos 2\theta + \frac{1}{3^4}\cos 3\theta + \ldots \infty$

(530) $\cos\theta - \frac{1}{2^4}\cos 2\theta + \frac{1}{3^4}\cos 3\theta - \ldots \infty$

(531) $\cos\theta + \frac{1}{3^4}\cos 3\theta + \frac{1}{5^4}\cos 5\theta + \ldots \infty$

(532) $\sin\theta - \frac{1}{3^4}\sin 3\theta + \frac{1}{5^4}\sin 5\theta - \ldots \infty$

(533) $\sin\theta + \frac{1}{2^5}\sin 2\theta + \frac{1}{3^5}\sin 3\theta + \ldots \infty$

(534) $\sum_{-\infty}^{+\infty} \frac{\cos n\theta}{n-a}$

(535) $\sum_{-\infty}^{+\infty} \frac{\sin n\theta}{n-a}$

(536) $a\cos\theta + \frac{a^2}{2}\cos 2\theta + \frac{a^3}{3}\cos 3\theta + \ldots \infty$

(537) $a\cos\theta + \frac{a^3}{3}\cos 3\theta + \frac{a^5}{5}\cos 5\theta + \ldots \infty$

Reference

$$= \frac{1}{12}\{(\theta - \pi)^3 - \pi^2\theta + \pi^3\} \quad \text{where } 0 \leqslant \theta \leqslant 2\pi$$ A. 362

$$= \frac{1}{12}(\pi^2\theta - \theta^3) \quad \text{where } -\pi \leqslant \theta \leqslant \pi$$ A. 362

$$= \frac{1}{8}(\pi^2\theta - \pi\theta^2) \quad \text{where } 0 \leqslant \theta \leqslant \pi$$

$$= \frac{1}{8}(\pi^2\theta + \pi\theta^2) \quad \text{where } -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$$ A. 362

$$= \frac{1}{8}\pi\left(\frac{1}{4}\pi^2 - \theta^2\right) \quad \text{where } -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$$ A. 362

$$= \frac{1}{48}\{2\pi^2(\theta - \pi)^2 - (\theta - \pi)^4\} - \frac{7\pi^4}{720} \quad \text{where } 0 \leqslant \theta \leqslant 2\pi$$ A. 363

$$= \frac{1}{48}\left\{\theta^4 - 2\pi^2\theta^2 + \frac{7}{15}\pi^4\right\} \quad \text{where } -\pi \leqslant \theta \leqslant \pi$$ A. 363

$$= \frac{\pi}{96}\{4\theta^3 - 6\pi\theta^2 + \pi^3\} \quad \text{where } 0 \leqslant \theta \leqslant \pi$$ A. 363

$$= \frac{\pi\theta}{96}\{3\pi^2 - 4\theta^2\} \quad \text{where } -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$$ A. 363

$$= \frac{\pi^4\theta}{90} - \frac{\pi^2\theta^3}{36} + \frac{\pi\theta^4}{48} - \frac{\theta^5}{240} \quad \text{where } 0 < \theta < 2\pi$$ T. 138

$$= -\frac{\pi\cos a(\pi - \theta)}{\sin a\pi} \quad \text{where } 0 < \theta < 2\pi$$ A. 370

$$= \frac{\pi\sin a(\pi - \theta)}{\sin a\pi} \quad \text{where } 0 < \theta < 2\pi$$ A. 370

$$= -\frac{1}{2}\operatorname{logh}(1 - 2a\cos\theta + a^2) \quad \text{where } a^2 < 1 \text{ and } \theta \neq 2n\pi$$ 2Z. 302

$$= \frac{1}{4}\operatorname{logh}\frac{1 + 2a\cos\theta + a^2}{1 - 2a\cos\theta + a^2} \quad \text{where } |a| < 1$$ E. 122

Series No.

(538) $\operatorname{logh} a^2 - 2\left(\frac{1}{a}\cos\theta + \frac{1}{2a^2}\cos 2\theta + \frac{1}{3a^3}\cos 3\theta + \ldots \infty\right)$

(539) $a\cos\theta - \frac{a^3}{3}\cos 3\theta + \frac{a^5}{5}\cos 5\theta - \ldots \infty$

(540) $a\sin\theta + \frac{a^2}{2}\sin 2\theta + \frac{a^3}{3}\sin 3\theta + \ldots \infty$

(541) $(\pi - \theta) - \left(\frac{1}{a}\sin\theta + \frac{1}{2a^2}\sin 2\theta + \frac{1}{3a^3}\sin 3\theta + \ldots \infty\right)$

(542) $a\sin\theta - \frac{a^2}{2}\sin 2\theta + \frac{a^3}{3}\sin 3\theta - \ldots \infty$

(543) $a\sin\theta + \frac{a^3}{3}\sin 3\theta + \frac{a^5}{5}\sin 5\theta + \ldots \infty$

(544) $a\sin\theta - \frac{a^3}{3}\sin 3\theta + \frac{a^5}{5}\sin 5\theta - \ldots \infty$

(545) $\frac{1}{a}\cos\theta + \frac{1}{a^2}\cos 2\theta + \frac{1}{a^3}\cos 3\theta + \ldots \infty$

(546) $\sum_{1}^{\infty} \frac{a^{2n-1}}{2n-1}\sin(2n-1)\theta$

(547) $\sum_{1}^{\infty} (-1)^{n-1}\frac{a^{2n-1}}{2n-1}\cos(2n-1)\theta$

(548) $2a\sin\theta + 4a^2\sin 2\theta + 6a^3\sin 3\theta + \ldots \infty$

(549) $\sum_{0}^{\infty} \frac{\cos(n+\frac{1}{2})\theta}{n+\frac{1}{2}}$

(550) $\sum_{0}^{\infty} \frac{\sin(n+\frac{1}{2})\theta}{n+\frac{1}{2}}$

(551) $\sum_{-\infty}^{+\infty} \frac{\sin(n-a)\theta}{n-a}$

		Reference
$= \operatorname{logh}(1 - 2a\cos\theta + a^2)$	where $a^2 > 1$	2Z. 302
$= \dfrac{1}{2}\tan^{-1}\dfrac{2a\cos\theta}{1 - a^2}$	where $\lvert a\rvert < 1$	E. 122
$= \tan^{-1}\dfrac{a\sin\theta}{1 - a\cos\theta}$	where $a^2 < 1$	2Z. 302
$= \tan^{-1}\dfrac{a\sin\theta}{1 - a\cos\theta}$	where $a^2 > 1$	2Z. 302
$= \tan^{-1}\dfrac{a\sin\theta}{1 + a\cos\theta}$	where $\lvert a\rvert < 1, \quad \theta \neq (2n + 1)\pi$	E. 122
$= \dfrac{1}{2}\tan^{-1}\dfrac{2a\sin\theta}{1 - a^2}$	where $\lvert a\rvert < 1, \quad \theta \neq n\pi$	E. 122
$= \dfrac{1}{4}\operatorname{logh}\dfrac{1 + 2a\sin\theta + a^2}{1 - 2a\sin\theta + a^2}$	where $\lvert a\rvert < 1$	E. 122
$= \dfrac{a\cos\theta - 1}{1 - 2a\cos\theta + a^2}$	where $a^2 > 1$	2Z. 302
$= \dfrac{1}{2}\tan^{-1}\dfrac{2a\sin\theta}{1 - a^2}$	where $a^2 < 1$	T. 139
$= \dfrac{1}{2}\tan^{-1}\dfrac{2a\cos\theta}{1 - a^2}$	where $a^2 < 1$	T. 140
$= \dfrac{2a(1 - a^2)\sin\theta}{(1 - 2a\cos\theta + a^2)^2}$	where $a^2 < 1$	2Z. 303
$= \operatorname{logh}\cot\dfrac{\theta}{4}$	where $0 < \theta < \pi$	A. 392
$= \dfrac{\pi}{2}$	where $0 < \theta < \pi$	A. 392
$= \pi$	where $0 < \theta < 2\pi$	A. 371

Series No.

(552) $\sum_{-\infty}^{+\infty} \frac{\cos (n-a)\theta}{n-a}$

(553) $2m \sum_{1}^{\infty} \frac{\cos n\theta}{n^2 - m^2}$

(554) $2 \sum_{1}^{\infty} \frac{n \sin n\theta}{n^2 - m^2}$

(555) $\sum_{1}^{\infty} \frac{\cos n\theta}{n^2 - a^2}$

(556) $\sum_{1}^{\infty} \frac{n \sin n\theta}{n^2 - a^2}$

(557) $\sum_{1}^{\infty} (-1)^{n-1} \frac{n \sin n\theta}{n^2 - a^2}$

(558) $a \sum_{1}^{\infty} (-1)^{n-1} \frac{\cos n\theta}{n^2 - a^2}$

(559) $2a \sum_{1}^{\infty} \frac{\cos n\theta}{n^2 + a^2}$

(560) $2 \sum_{1}^{\infty} \frac{n \sin n\theta}{n^2 + a^2}$

(561) $\sum_{1}^{\infty} (-1)^{n-1} \frac{n \sin n\theta}{n^2 + a^2}$

(562) $a \sum_{1}^{\infty} (-1)^{n} \frac{\cos n\theta}{n^2 + a^2}$

(563) $\sum_{2}^{\infty} \frac{(-1)^n}{n} \left[\frac{\cos 2(n-1)\theta}{(n-1)^2} - \frac{\cos 2(n+1)\theta}{(n+1)^2}\right]$

Reference

$= -\pi \cot \pi a \quad \text{where } 0 < \theta < 2\pi$ A. 371

$= \frac{1}{m} + \frac{\cos m\theta}{2m} - (\pi - \theta) \sin m\theta \quad \text{where } 0 < \theta < 2\pi \text{ and } m \text{ is a positive integer (omit } m = n)$ A. 371

$= (\pi - \theta) \cos m\theta - \frac{\sin m\theta}{2m} \quad \text{where } 0 < \theta < 2\pi \text{ and } m \text{ is a positive integer (omit } m = n)$ A. 371

$= \frac{\pi \sin a(\frac{1}{2}\pi - \theta)}{4a \cos \frac{1}{2}\pi a} \quad \text{where } 0 < \theta < \pi \text{ and } n \text{ is odd (} a \text{ unrestricted)}$ A. 371

$= \frac{\pi \cos a(\frac{1}{2}\pi - \theta)}{4 \cos \frac{1}{2}\pi a} \quad \text{where } 0 < \theta < \pi \text{ and } n \text{ is odd (} a \text{ unrestricted)}$ A. 371

$= \frac{\pi \sin a\theta}{2 \sin \pi a} \quad \text{where } -\pi < \theta < \pi$ Q. 191

$= \frac{\pi \cos a\theta}{2 \sin \pi a} - \frac{1}{2a} \quad \text{where } -\pi < \theta < \pi$ Q. 191

$= \frac{\pi \cosh a(\pi - \theta)}{\sinh \pi a} - \frac{1}{a} \quad \text{where } 0 < \theta < 2\pi$ {A. 393, AE. 5}

$= \frac{\pi \sinh a(\pi - \theta)}{\sinh \pi a} \quad \text{where } 0 < \theta < 2\pi$ {A. 393, AE. 4}

$= \frac{\pi \sinh a\theta}{2 \sinh \pi a} \quad \text{where } -\pi < \theta < \pi$ 2Z. 717

$= + \frac{\pi \cosh a\theta}{2 \sinh \pi a} - \frac{1}{2a} \quad \text{where } -\pi < \theta < \pi$ 2Z. 717

$= 1 + \frac{\pi^2}{6} - 2\theta^2 - 2 \cos 2\theta - 2\theta \sin 2\theta - \frac{1}{4} \cos 4\theta$

Series No.

(564) $\frac{a}{\pi}\sum_{1}^{\infty}\frac{\sin\frac{2n\pi\theta}{a}}{n}$

See No. (572)

(565) $\frac{a^2}{\pi^2}\sum_{1}^{\infty}\frac{\cos\frac{2n\pi\theta}{a}}{n^2}$

See No. (573)

(566) $\frac{3}{4}\sin\theta + 3\sin^3\frac{\theta}{3} + 3^2\sin^3\frac{\theta}{3^2} + \ldots \infty$

(567) $\sin^2\theta + 2^2\sin^4\frac{\theta}{2} + 2^4\sin^4\frac{\theta}{2^2} + 2^6\sin^4\frac{\theta}{2^3} + \ldots \infty$

(568) $\cos 2\theta - \frac{1}{2}\cos 4\theta + \frac{1}{3}\cos 6\theta - \frac{1}{4}\cos 8\theta + \frac{1}{5}\cos 10\theta + \ldots \infty$

(569) $1 - \frac{1}{4}\tan^2 2\theta + \frac{1}{8}\tan^4 2\theta - \frac{5}{64}\tan^6 2\theta + \ldots \infty$

(570) $1 - \frac{3}{8}\tan^2 2\theta + \frac{31}{128}\tan^4 2\theta - \frac{187}{1024}\tan^6 2\theta + \ldots \infty$

(571) $1 - \frac{11}{32}\tan^2 2\theta + \frac{431}{2048}\tan^4 2\theta - \ldots \infty$

(572)† $2\sum_{1}^{\infty}\frac{\sin 2n\pi\theta}{2n\pi}$

(573)† $2\sum_{1}^{\infty}\frac{\cos 2n\pi\theta}{(2n\pi)^2}$

(574)† $2\sum_{1}^{\infty}\frac{\sin 2n\pi\theta}{(2n\pi)^{2k-1}}$

(575)† $2\sum_{1}^{\infty}\frac{\cos 2n\pi\theta}{(2n\pi)^{2k}}$

† For $\phi_k(\theta)$ see No. (1128). B_k are Bernoulli numbers.

Reference

$$= \frac{1}{2}a - \theta \quad \text{where } 0 < \theta < a$$

$$= \left(\frac{1}{2}a - \theta\right)^2 - \frac{a^2}{12} \quad \text{where } 0 \leqslant \theta \leqslant a$$

$$= \frac{3}{4}\theta$$ C. 336

$$= \theta^2$$ C. 336

$$= \text{logh } 2 + \text{logh} \cos\theta \quad \text{where } -\pi < \theta < \pi$$ 2Z. 302

$$= \frac{2\tan\theta}{\tan 2\theta}$$ O. 383

$$= \frac{2\sin\theta}{\tan 2\theta}$$ O. 383

$$= \frac{4\sin\frac{\theta}{2}}{\tan 2\theta}$$ O. 383

$$= \frac{1}{2} - \phi_1(\theta) = \frac{1}{2} - \theta \quad \text{where } 0 < \theta < 1$$ {A. 370, AE. 2}

$$= \frac{1}{2!}\{\phi_2(\theta) + B_1\} = \frac{1}{2}\left(\theta^2 - \theta + \frac{1}{6}\right) \quad \text{where } 0 < \theta < 1$$ {A. 370, AE. 2}

$$= \frac{(-1)^k}{(2k-1)!}\phi_{2k-1}(\theta) \quad \text{where } k > 1 \text{ and } 0 < \theta < 1$$ A. 370

$$= \frac{(-1)^{k-1}}{(2k)!}\{\phi_{2k}(\theta) + (-1)^{k-1}B_k\} \quad \text{where } 0 < \theta < 1$$ A. 370

Series No.

(576)† $\sin 2\pi\theta + \frac{\sin 4\pi\theta}{2^{2n+1}} + \frac{\sin 6\pi\theta}{3^{2n+1}} + \ldots \infty$

(577)† $\cos 2\pi\theta + \frac{\cos 4\pi\theta}{2^{2n}} + \frac{\cos 6\pi\theta}{3^{2n}} + \ldots \infty$

(578) $\sum_1^\infty \tan^{-1} \frac{2\theta^2}{n^2}$

(579) $\sum_1^\infty (-1)^{n-1} \tan^{-1} \frac{2\theta^2}{n^2}$

(580) $\tan^{-1} \frac{\theta}{1-\theta^2} - \tan^{-1} \frac{3\theta}{3^2-\theta^2} + \tan^{-1} \frac{5\theta}{5^2-\theta^2} - \ldots \infty$

(581) $\frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{2^2}\tan\frac{\theta}{2^2} + \frac{1}{2^3}\tan\frac{\theta}{2^3} + \ldots \infty$

(582) $\frac{1}{2^2}\sec^2\frac{\theta}{2} + \frac{1}{2^4}\sec^2\frac{\theta}{2^2} + \frac{1}{2^6}\sec^2\frac{\theta}{2^3} + \ldots \infty$

(583) $\left\{\frac{\pi^2}{1} - \frac{4}{1^3}\right\}\sin\theta - \frac{\pi^2}{2}\sin 2\theta + \left\{\frac{\pi^2}{3} - \frac{4}{3^3}\right\}\sin 2\theta - \frac{\pi^2}{4}\sin 4\theta + \ldots \infty$

(584) $\left\{\frac{\pi^3}{1} - \frac{6\pi}{1^3}\right\}\sin\theta - \left\{\frac{\pi^3}{2} - \frac{6\pi}{2^3}\right\}\sin 2\theta + \left\{\frac{\pi^3}{3} - \frac{6\pi}{3^3}\right\}\sin 3\theta - \ldots \infty$

(585) $\frac{1}{2}(1+\epsilon^\pi)\sin\theta + \frac{2}{5}(1-\epsilon^\pi)\sin 2\theta + \frac{3}{10}(1+\epsilon^\pi)\sin 3\theta + \frac{4}{17}(1-\epsilon^\pi)\sin 4\theta - \ldots \infty$

(586) $\left\{\frac{\pi^2}{1^2} - \frac{4}{1^4}\right\}\cos\theta - \frac{\pi^2}{2^2}\cos 2\theta + \left\{\frac{\pi^2}{3^2} - \frac{4}{3^4}\right\}\cos 3\theta - \frac{\pi^2}{4^2}\cos 4\theta + \left\{\frac{\pi^2}{5^2} - \frac{4}{5^4}\right\}\cos 5\theta - \ldots \infty$

† For values of $B_n(\theta)$, see No. (1142).

Reference

$$= (-1)^{n+1} \frac{2^{2n}\pi^{2n+1}}{(2n)!} B_{2n+1}(\theta) \quad \text{where } 0 < \theta < 1$$ AE. 3

$$= (-1)^{n-1} \frac{2^{2n-1}\pi^{2n}}{(2n-1)!} \left\{ B_{2n}(\theta) + (-1)^{n-1} \frac{B_n}{2n} \right\} \quad \text{where } 0 < \theta < 1$$ AE. 3

$$= \frac{\pi}{4} - \tan^{-1} \frac{\tanh \pi\theta}{\tan \pi\theta}$$ A. 314

$$= -\frac{\pi}{4} + \tan^{-1} \frac{\sinh \pi\theta}{\sin \pi\theta}$$ A. 314

$$= \tan^{-1} \frac{\sinh \frac{\pi\theta}{4}}{\cos \frac{\sqrt{3}\pi\theta}{4}}$$ A. 314

$$= \frac{1}{\theta} - \cot \theta$$ Y. 54

$$= \operatorname{cosec}^2 \theta - \frac{1}{\theta^2}$$ Y. 54

$$= \frac{\pi\theta^2}{2}$$ U. 41

$$= \frac{\pi\theta^3}{2}$$ U. 41

$$= \frac{\pi\epsilon^\theta}{2}$$ U. 42

$$= \left\{ \frac{\pi^3}{4} - \theta^3 \right\} \frac{\pi}{6}$$ U. 45

Series No.

(587) $\epsilon^a \cos\theta - \frac{1}{3}\epsilon^{3a}\cos 3\theta + \frac{1}{5}\epsilon^{5a}\cos 5\theta - \dots \infty$

(588) $\frac{8a}{\pi^2}\sum_0^\infty \left\{\frac{\cos(2n+1)\frac{\pi\theta}{2a}}{2n+1}\right\}^2$

(589) $\sum_1^\infty \frac{n}{n^2+a^2}\{1 + (-1)^{n-1}\epsilon^{a\pi}\}\sin n\theta$

(590) $1 + a^2 + 4a\cos\theta + 2a^2(3 - a^2)\cos 2\theta$
$+ 2a^3(4 - 2a^2)\cos 3\theta + \dots$
$+ 2a^n\{n(1 - a^2) + (1 + a^2)\}\cos n\theta + \dots \infty$

(591) $a^2 + 1 + 4a\cos\theta + \frac{2(3a^2 - 1)}{a^2}\cos 2\theta + \dots$
$+ \frac{2\{n(a^2 - 1) + (a^2 + 1)\}}{a^n}\cos n\theta + \dots \infty$

(592) $\sum_1^\infty \frac{na^{n-1}}{2(1 - a^2)^3}[n(1 - a^2) + (1 + a^2)]\sin\theta$

(593) $\sum_1^\infty \frac{n}{2(a^2 - 1)^3}\frac{1}{a^{n+1}}[n(a^2 - 1) + (a^2 + 1)]\sin n\theta$

Memorandum: Consult Edwards *Integral Calculus*, Vol. II, for extension to these series.

(594) $\frac{\epsilon^{-q\theta}\sin\theta}{1} + \frac{\epsilon^{-2q\theta}\sin 2\theta}{2} + \frac{\epsilon^{-3q\theta}\sin 3\theta}{3} + \dots \infty$

(595) $\frac{\epsilon^{-q\theta}\cos\theta}{1} + \frac{\epsilon^{-2q\theta}\cos 2\theta}{2} + \frac{\epsilon^{-3q\theta}\cos 3\theta}{3} + \dots \infty$

(596) $\frac{4}{3}\sin 2\theta + \frac{8}{15}\sin 4\theta + \dots + \frac{1 + (-1)^n}{n^2 - 1}n\sin n\theta + \dots \infty$

Reference

$= -\frac{1}{2}\tan^{-1}(\cos\theta \operatorname{cosech} a)$ E. 123

$= a - \theta$ where θ is positive, $|\theta| < a$

$= a + \theta$ where θ is negative, $|\theta| < a$ J. 309

$= \frac{\pi}{2}\epsilon^{a\theta}$ where $0 < \theta < \pi$ U. 42

$= \frac{(1-a^2)^3}{(1-2a\cos\theta+a^2)^2}$ where $a^2 < 1$ 2Z. 303

$= \frac{(a^2-1)^3}{(1-2a\cos\theta+a^2)^2}$ where $a^2 > 1$ 2Z. 304

$= \frac{\sin\theta}{(1-2a\cos\theta+a^2)^3}$ where $a^2 < 1$ 2Z. 305

$= \frac{\sin\theta}{(1-2a\cos\theta+a^2)^3}$ where $a^2 > 1$ 2Z. 305

$= \tan^{-1}\frac{\sin\theta}{\epsilon^{q\theta}-\cos\theta}$ where q is positive 2Z. 194

$= -\operatorname{logh}\sqrt{1-2\epsilon^{-q\theta}\cos\theta+\epsilon^{-2q\theta}}$ where q is positive 2Z. 194

$= \frac{\pi\cos\theta}{2}$ where $0 < \theta < \pi$

Series No.

(597) $\frac{1}{2}(1+\cosh\pi)\sin\theta+\frac{2}{5}(1-\cosh\pi)\sin 2\theta$
$+\frac{3}{10}(1+\cosh\pi)\sin 3\theta+\ldots\infty$

(598) $\frac{1+\epsilon^a}{a^2+\pi^2}\sin\frac{\pi\theta}{a}+\frac{2(1-\epsilon^a)}{a^2+4\pi^2}\sin\frac{2\pi a}{a}$
$+\frac{3(1+\epsilon^a)}{a^2+9\pi^2}\sin\frac{3\pi\theta}{a}+\ldots\infty$

(599) $\sum_1^\infty \frac{(-1)^{n-1}}{n^2}\cos\frac{n\pi\theta}{a}$

(600) $\sum_1^\infty \frac{\cos 2n\theta}{(2n-1)(2n+1)}$

(601) $\frac{6}{\pi}\sum_1^\infty \frac{\sin\frac{1}{3}(2n-1)\pi\cdot\sin(2n-1)\theta}{(2n-1)^2}$

(602) $\sum_1^\infty \frac{\cos(2n+2)\theta}{n(n+1)}$

(603) $\sum_1^\infty \frac{\sin(2n+2)\theta}{n(n+1)}$

(604) $\frac{\sin 2\theta}{1\cdot 3}+\frac{\sin 3\theta}{2\cdot 4}+\ldots\infty=\sum_2^\infty \frac{\sin n\theta}{n^2-1}$

(605) $\frac{\cos 2\theta}{1\cdot 3}+\frac{\cos 3\theta}{2\cdot 4}+\ldots\infty=\sum_2^\infty \frac{\cos n\theta}{n^2-1}$

(606) $\frac{\cos 2\theta}{1\cdot 3}-\frac{\cos 4\theta}{3\cdot 5}+\ldots\infty=-\sum_2^\infty \frac{\cos\frac{n\pi}{2}\cos n\theta}{n^2-1}$

Reference

$$= \frac{\pi \cosh \theta}{2}$$ U. 42

$$= \frac{\epsilon^{\theta}}{2\pi} \quad \text{where } 0 < \theta < a$$ U. 51

$$= \frac{\pi^2}{4a^2}\left(\frac{a^2}{3} - \theta^2\right) \quad \text{where } -a \leqslant \theta \leqslant a$$ T. 139

$$= \frac{1}{2} - \frac{\pi}{4}\sin\theta \quad \text{where } 0 \leqslant \theta \leqslant \frac{\pi}{2}$$ T. 139

$$= \frac{3}{2}\theta \quad \text{where } 0 \leqslant \theta \leqslant \frac{1}{3}\pi$$

$$= \frac{\pi}{2} \quad \text{where } \frac{1}{3}\pi \leqslant \theta \leqslant \frac{2}{3}\pi$$ AB. 243

$$= \frac{3}{2}(\pi - \theta) \quad \text{where } \frac{2}{3}\pi \leqslant \theta \leqslant \pi$$

$$= \cos 2\theta - \left(\frac{\pi}{2} - \theta\right)\sin 2\theta + \sin^2\theta \operatorname{logh}(4\sin^2\theta)$$
$$\text{where } 0 \leqslant \theta \leqslant \pi$$ T. 139

$$= \sin 2\theta - (\pi - 2\theta)\sin^2\theta - \sin\theta\cos\theta \operatorname{logh}(4\sin^2\theta)$$
$$\text{where } 0 \leqslant \theta \leqslant \pi$$ T. 139

$$= \frac{1}{4}\sin\theta - \sin\theta \operatorname{logh}\left(2\sin\frac{\theta}{2}\right) \quad \text{where } 0 < \theta < 2\pi$$ A. 368

$$= \frac{1}{2} + \frac{1}{4}\cos\theta - \frac{1}{2}(\pi - \theta)\sin\theta \quad \text{where } 0 < \theta < 2\pi$$ A. 368

$$= \frac{\pi}{4}\cos\theta - \frac{1}{2}$$

Series No.

(607) $\dfrac{2\sin 2\theta}{1\cdot 3} - \dfrac{4\sin 4\theta}{3\cdot 5} + \ldots \infty = -\sum_{2}^{\infty} \dfrac{\cos\frac{n\pi}{2}\, n\sin n\theta}{n^2-1}$

(608) $\dfrac{\cos 2\theta}{1\cdot 3} - \dfrac{\cos 3\theta}{2\cdot 4} + \ldots \infty = +\sum_{2}^{\infty} \dfrac{(-1)^n \cos n\theta}{n^2-1}$

(609) $\dfrac{1}{2} - \dfrac{\cos 2\theta}{1\cdot 3} - \dfrac{\cos 4\theta}{3\cdot 5} - \dfrac{\cos 6\theta}{5\cdot 7} - \ldots \infty$

(610) $\dfrac{1}{2} + \dfrac{1}{4}\cos\theta - \dfrac{1}{1\cdot 3}\cos 2\theta - \dfrac{1}{2\cdot 4}\cos 3\theta - \dfrac{1}{3\cdot 5}\cos 4\theta - \ldots \infty$

(611) $\dfrac{\cos 3\theta}{2\cdot 4} + \dfrac{\cos 5\theta}{4\cdot 6} + \ldots \infty = \sum_{2}^{\infty} \dfrac{\sin^2\frac{n\pi}{2}\cos n\theta}{n^2-1}$

(612) $\dfrac{\sin 3\theta}{2\cdot 4} - \dfrac{\sin 5\theta}{4\cdot 6} + \ldots \infty = -\sum_{2}^{\infty} \dfrac{\sin\frac{n\pi}{2}\sin n\theta}{n^2-1}$

(613) $\dfrac{\sin 3\theta}{2\cdot 4} + \dfrac{\sin 5\theta}{4\cdot 6} + \ldots \infty = \sum_{2}^{\infty} \dfrac{\sin^2\frac{n\pi}{2}\sin n\theta}{n^2-1}$

(614) $\dfrac{3\cos 3\theta}{2\cdot 4} - \dfrac{5\cos 5\theta}{4\cdot 6} + \ldots \infty = -\sum_{2}^{\infty} \dfrac{\sin\frac{n\pi}{2}\, n\cos n\theta}{n^2-1}$

(615) $\dfrac{2\cos 2\theta}{1\cdot 3} + \dfrac{3\cos 3\theta}{2\cdot 4} + \ldots \infty = \sum_{2}^{\infty} \dfrac{n\cos n\theta}{n^2-1}$

(616) $\sum_{0}^{\infty} \dfrac{(-1)^n \cos n\theta}{(n+1)(n+2)}$

(617) $\dfrac{1}{2}\sin\theta + \dfrac{1\cdot 3}{2\cdot 4}\sin 2\theta + \dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\sin 3\theta + \ldots \infty$

Reference

$$= \frac{\pi}{4} \sin\theta$$

$$= \frac{1}{2} - \frac{1}{4}\cos\theta - \frac{1}{2}\theta\sin\theta$$

$$= \frac{\pi}{4} \sin\theta$$ T. 139

$$= \frac{1}{2}(\pi - \theta)\sin\theta \quad \text{where } 0 \leqslant \theta \leqslant \pi$$ Q. 166

$$= \frac{1}{2}(\pi - \theta)\sin\theta \quad \text{where } \pi \leqslant \theta \leqslant 2\pi$$

$$= -\frac{1}{2}(\pi + \theta)\sin\theta \quad \text{where } -\pi \leqslant \theta \leqslant 0$$

$$= \frac{1}{4}\cos\theta - \frac{\pi}{4}\sin\theta + \frac{1}{2}\theta\sin\theta$$

$$= \frac{1}{2}\theta\cos\theta - \frac{1}{4}\sin\theta$$

$$= \frac{1}{4}\sin\theta - \frac{1}{2}\sin\theta \operatorname{logh}(2\sin\theta)$$

$$= -\frac{1}{2}\theta\sin\theta + \frac{1}{4}\cos\theta$$

$$= -\frac{1}{2} - \frac{1}{4}\cos\theta - \cos\theta \operatorname{logh}\left(2\sin\frac{\theta}{2}\right)$$

$$= (\cos\theta + \cos 2\theta)\operatorname{logh}\left(2\cos\frac{\theta}{2}\right) + \frac{1}{2}\theta(\sin 2\theta + \sin\theta) - \cos\theta$$

Q. 190

$$= \left(2\sin\frac{\theta}{2}\right)^{-1/2} \sin\frac{\pi - \theta}{4}$$ E. 116

Series No.

(618) $1 + \frac{1}{2}\cos\theta + \frac{1\cdot 3}{2\cdot 4}\cos 2\theta + \ldots\ \infty$

(619) $\frac{\sin 2\theta}{1\cdot 2\cdot 3} + \frac{\sin 3\theta}{2\cdot 3\cdot 4} + \ldots\ \infty = \sum_{2}^{\infty} \frac{\sin n\theta}{(n-1)n(n+1)}$

(620) $\cos\theta + \frac{1}{2}\cdot\frac{\cos 3\theta}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{\cos 5\theta}{5} + \ldots\ \infty$

(621) $\sin\theta + \frac{1}{2}\sin 3\theta + \frac{1\cdot 3}{2\cdot 4}\cdot\sin 5\theta + \ldots\ \infty$

(622) $1 + \frac{1}{2}\cos 2\theta - \frac{1}{2\cdot 4}\cos 4\theta + \frac{1\cdot 3}{2\cdot 4\cdot 6}\cos 6\theta - \ldots\ \infty$

(623) $\frac{\sin\theta}{3} - \frac{\sin 3\theta}{1\cdot 3\cdot 5} - \frac{\sin 5\theta}{3\cdot 5\cdot 7} - \ldots\ \infty = -\sum_{1}^{\infty} \frac{\sin n\theta}{n(n^2-4)}$

(624) $1 - \frac{\sin^2\theta}{3} - \frac{2}{3}\cdot\frac{\sin^4\theta}{5} - \frac{2\cdot 4}{3\cdot 5}\frac{\sin^6\theta}{7} - \ldots\ \infty$

(625) $\sin\theta + \frac{2}{3}\sin^3\theta + \frac{2\cdot 4}{3\cdot 5}\sin^5\theta + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\sin^7\theta + \ldots\ \infty$

(626) $\frac{\cos\theta}{1\cdot 2\cdot 3} + \frac{\cos 2\theta}{2\cdot 3\cdot 4} + \frac{\cos 3\theta}{3\cdot 4\cdot 5} + \ldots\ \infty$

(627) $\frac{\cos 3\theta}{1\cdot 3\cdot 5} - \frac{\cos 5\theta}{3\cdot 5\cdot 7} + \frac{\cos 7\theta}{5\cdot 7\cdot 9} - \ldots\ \infty$

(628) $\frac{1}{2}\cdot\frac{\sin^2\theta}{2} + \frac{1\cdot 3}{2\cdot 4}\frac{\sin^4\theta}{4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{\sin^6\theta}{6} + \ldots\ \infty$

(629) $1 + \frac{2}{3}\sin^2\theta + \frac{2\cdot 4}{3\cdot 5}\sin^4\theta + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\sin^6\theta + \ldots\ \infty$

$$= 1 + \frac{2^2}{3!}\sin^2\theta + \frac{2^2\cdot 4^2}{5!}\sin^4\theta + \frac{2^2\cdot 4^2\cdot 6^2}{7!}\sin^6\theta + \ldots\ \infty$$

(630) $\sin\theta + a\cos\theta - \frac{a^2}{2!}\sin\theta - \frac{a^3}{3!}\cos\theta + \ldots\ \infty$

(631) $1 + a\cos\theta + \frac{a^2}{2!}\cos 2\theta + \frac{a^3}{3!}\cos 3\theta + \ldots\infty$

	Reference
$= \left(2 \sin \frac{\theta}{2}\right)^{-1/2} \cos \frac{\pi - \theta}{4}$	E. 116
$= \frac{3}{4} \sin \theta - \frac{1}{2}(\pi - \theta) + \frac{1}{2}(\pi - \theta) \cos \theta$	
$= \frac{1}{2} \cos^{-1}(1 - 2 \sin \theta)$	C. 334
$= \frac{1}{\sqrt{2 \sin \theta}} \sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$	E. 117
$= \sqrt{\cos \theta(1 + \cos \theta)}$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	E. 118
$= \frac{\pi}{8} \sin^2 \theta$ where n is odd	X. 54
$= \theta \cot \theta$	Y. 86
$= \theta \sec \theta$	Y. 505
$= \frac{1}{2} \cos \theta - \frac{1}{4} \cos 2\theta$	C. 421
$= \frac{\pi}{8} \cos^2 \theta - \frac{1}{3} \cos \theta$ where $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	A. 369
$= \text{logh} \sec^2 \frac{\theta}{2}$	H. 498
$= \frac{2\theta}{\sin 2\theta}$	L. 78
$= \sin(\theta + a)$	Y. 84
$= \epsilon^{a \cos \theta} \cos(a \sin \theta)$	X. 72

Series No.

(632) $a \sin\theta + \frac{a^2}{2!}\sin 2\theta + \frac{a^3}{3!}\sin 3\theta + \ldots \infty$

(633) $1 + \frac{a^2 \cos 2\theta}{2!} + \frac{a^4 \cos 4\theta}{4!} + \ldots \infty$

(634) $\frac{a^2 \sin 2\theta}{2!} + \frac{a^4 \sin 4\theta}{4!} + \ldots \infty$

(635) $\sum_0^\infty \frac{a^{2n} \cos 2n\theta}{(2n)!}$

(636) $\sum_0^\infty \frac{a^{2n+1} \cos (2n+1)\theta}{(2n+1)!}$

(637) $\sum_0^\infty \frac{a^{2n+1} \sin (2n+1)\theta}{(2n+1)!}$

(638) $\sum_1^\infty \frac{a^{2n} \sin 2n\theta}{(2n)!}$

(639) $1 - \theta \sin\theta + \frac{\theta^2 \cos 2\theta}{2!} + \frac{2^2\theta^3 \sin 3\theta}{3!} - \frac{3^3\theta^4 \cos 4\theta}{4!} - \ldots \infty$

(640) $\theta \cos\theta + \frac{\theta^2 \sin 2\theta}{2!} - \frac{2^2\theta^3 \cos 3\theta}{3!} - \frac{3^3\theta^4 \sin 4\theta}{4!} + \ldots \infty$

(641) $r\theta \sin\phi + \frac{r^2\theta^2}{2!}\sin 2\phi + \frac{r^3\theta^3}{3!}\sin 3\phi + \ldots \infty$

$\left(r = \sqrt{a^2 + b^2},\right.$

(642) $1 + \sin\theta + \frac{(1 + 1^2)\sin^2\theta}{2!} + \frac{(1 + 2^2)\sin^3\theta}{3!} + \ldots \infty$

(643) $1 + 2^{1/2} \cos\frac{\pi}{4}\theta + 2^{2/2}\cos\frac{2\pi}{4}\frac{\theta^2}{2!} + 2^{3/2}\cos\frac{3\pi}{4}\frac{\theta^3}{3!} + \ldots \infty$

$$= \sum_0^\infty 2^{n/2} \cos\frac{n\pi}{4}\cdot\frac{\theta^n}{n!}$$

(644) $\cos\theta + n\cos 3\theta + \frac{n(n-1)}{2!}\cos 5\theta + \ldots \infty$

Reference

$= \epsilon^{a \cos \theta} \sin (a \sin \theta)$ X. 72

$= \frac{1}{2} \cos (a \sin \theta)\{\epsilon^{a \cos \theta} + \epsilon^{-a \cos \theta}\}$ E. 118

$= \frac{1}{2} \sin (a \sin \theta)\{\epsilon^{a \cos \theta} - \epsilon^{-a \cos \theta}\}$ E. 118

$= \cosh (a \cos \theta) \cos (a \sin \theta)$ where $a^2 < 1$ T. 127

$= \sinh (a \cos \theta) \cos (a \sin \theta)$ where $a^2 < 1$ T. 127

$= \cosh (a \cos \theta) \sin (a \sin \theta)$ where $a^2 < 1$ T. 127

$= \sinh (a \cos \theta) \sin (a \sin \theta)$ where $a^2 < 1$ T. 127

$$\left.\begin{array}{l} = \cos \theta \\ \\ = \sin \theta \end{array}\right\} \text{ where } |\theta| < \frac{1}{\epsilon}$$ A. 312

$\phi = \tan^{-1} \frac{b}{a}\Big) = \epsilon^{a\theta} \sin b\theta$ E. 131

$= \frac{\epsilon^{\theta}}{\cos \theta}$ L. 79

$= \epsilon^{\theta} \cos \theta$ Y. 84

$= 2^n \cos^n \theta \cos (n + 1)\theta$ X. 72

Series No.

(645) $\sin\theta + n\sin 3\theta + \frac{n(n-1)}{2!}\sin 5\theta + \ldots \infty$

(646) $\frac{1}{2} + \frac{a}{a+2}\cos 2\theta + \frac{a(a-2)}{(a+2)(a+4)}\cos 4\theta + \ldots \infty$

(647) $\cos^n\theta - \frac{n(n-1)}{2!}\cos^{n-2}\theta\sin^2\theta$
$+ \frac{n(n-1)(n-2)(n-3)}{4!}\cos^{n-4}\theta\sin^4\theta - \ldots$ zero

(648) $n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta\sin^3\theta + \ldots$ zero

(649) $\cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{2!}\cos(n-4)\theta + \ldots$

If n is odd there is an even number of terms so that the last term contains $\cos\theta$.

(650) $\cos n\theta - n\cos(n-2)\theta + \frac{n(n-1)}{2!}\cos(n-4)\theta + \ldots$
$+ \frac{1}{2}(-1)^{n/2}\frac{n!}{\left\{\left(\frac{n}{2}\right)!\right\}^2}$

(651) $\sin n\theta - n\sin(n-2)\theta + \frac{n(n-1)}{2!}\sin(n-4)\theta + \ldots$
$+ (-1)^{(n-1)/2}\frac{n!}{\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)!}\sin\theta$

In the series Nos. (652) through (659) ${}_nC_r$ are the binomial coefficients.

$${}_nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$$

(652) $\cos 2n\theta + {}_{2n}C_1\cos(2n-2)\theta + {}_{2n}C_2\cos(2n-4)\theta + \ldots$
$+ \frac{1}{2}\,{}_{2n}C_n$

(653) $\cos(2n+1)\theta + {}_{2n+1}C_1\cos(2n-1)\theta + \ldots + {}_{2n+1}C_n\cos\theta$

Reference

$= 2^n \cos^n \theta \sin (n + 1)\theta$ X. 72

$= \dfrac{2^{a-1}\left\{\Gamma\left(\dfrac{a}{2} + 1\right)\right\}^2}{\Gamma(a + 1)} \cos^a\theta$ where a is positive but not necessarily an integer and $-\dfrac{\pi}{2} \leqslant \theta \leqslant \dfrac{\pi}{2}$ Q. 263

$= \cos n\theta$ E. 33

$= \sin n\theta$ E. 33

$= 2^{n-1} \cos^n \theta$ E. 55

$= 2^{n-1}(-1)^{n/2} \sin^n \theta$ where n is even E. 57

$= 2^{n-1}(-1)^{(n-1)/2} \sin^n \theta$ where n is odd E. 58

$= 2^{2n-1} \cos^{2n} \theta$ where n is even C. 278

$= 2^{2n} \cos^{2n+1} \theta$ C. 278

Series No.

(654) $\cos 2n\theta - {}_{2n}C_1 \cos(2n-2)\theta + {}_{2n}C_2 \cos(2n-4)\theta + \dots + (-1)^n \frac{1}{2}\cdot {}_{2n}C_n$

(655) $\sin(2n+1)\theta - {}_{2n+1}C_1 \sin(2n-1)\theta + {}_{2n+1}C_2 \sin(2n-3)\theta + \dots + (-1)^n {}_{2n+1}C_n \sin\theta$

(656) $1 + {}_nC_1 \cos 2\theta + {}_nC_2 \cos 4\theta + {}_nC_3 \cos 6\theta + \dots + \cos 2n\theta$

(657) ${}_nC_1 \sin 2\theta + {}_nC_2 \sin 4\theta + {}_nC_3 \sin 6\theta + \dots + \sin 2n\theta$

(658) $1 - {}_nC_1 \cos 2\theta + {}_nC_2 \cos 4\theta - {}_nC_3 \cos 6\theta + \dots - \cos 2n\theta$

${}_nC_1 \sin 2\theta - {}_nC_2 \sin 4\theta + {}_nC_3 \sin 6\theta - \dots - \sin 2n\theta$

(659) ${}_nC_1 \sin 2\theta - {}_nC_2 \sin 4\theta + {}_nC_3 \sin 6\theta - \dots + \sin 2n\theta$

$1 - {}_nC_1 \cos 2\theta + {}_nC_2 \cos 4\theta - {}_nC_3 \cos 6\theta + \dots + \cos 2n\theta$

(660) $1 - a\theta \sin b\theta - \frac{a(a-2b)}{2!}\theta^2 \cos 2b\theta + \frac{a(a-3b)^2}{3!}\theta^3 \sin 3b\theta + \dots \infty$

(661) $\frac{1}{2} + \frac{2n-1}{2n+1}\cos 2\theta + \frac{(2n-1)(2n-3)}{(2n+1)(2n+3)}\cos 4\theta + \dots \infty$

(662) $\sin^2\theta + \frac{2}{3}\cdot\frac{1}{2}\sin^4\theta + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{1}{3}\sin^6\theta + \dots \infty$

(663) $\cos\theta - \frac{1}{7}\cos 3\theta + \frac{1}{7}\cdot\frac{5}{11}\cos 5\theta - \frac{1}{7}\cdot\frac{5}{11}\cdot\frac{9}{15}\cos 7\theta + \dots \infty$

$$\text{where } h = \frac{4}{\pi}\int_0^{\pi/2} \cos^{3/2}\theta \, d\theta$$

(664) General case of (663)

$$\cos\theta - \frac{1-n}{3+n}\cos 3\theta + \frac{1-n}{3+n}\cdot\frac{3-n}{5+n}\cos 5\theta - \frac{1-n}{3+n}\cdot\frac{3-n}{5+n}\cdot\frac{5-n}{7+n}\cos 7\theta + \dots \infty$$

When n is odd, this series terminates.

Reference

$= 2^{2n-1}(-1)^n \sin^{2n}\theta$ C. 278

$= 2^{2n}(-1)^n \sin^{2n+1}\theta$ C. 278

$= 2^{\pi}\cos n\theta \cos^n\theta$ where n can be odd or even

$= 2^n \cos^n\theta \sin n\theta$ where n can be odd or even

$= (-1)^{(n-1)/2}2^n \sin^n\theta \sin n\theta$ where n is odd

$= (-1)^{(n+2)/2}2^n \sin^n\theta \sin n\theta$ where n is even

$= (-1)^{(n-1)/2}2^n \sin^n\theta \cos n\theta$ where n is odd

$= (-1)^{n/2}2^n \sin^n\theta \cos n\theta$ where n is even

$= \cos a\theta$ Y. 511

$= \frac{\pi}{4}\cdot\frac{1\cdot3\cdot5\ldots(2n-1)}{2\cdot4\cdot6\ldots(2n-2)}|\cos^{2n-1}\theta|$ where n is an integer Q. 191

$= \theta^2$ Q. 130

$= \frac{1}{h}\cos^{1/2}\theta$

$= \frac{4}{\pi}\cdot\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{7}{4}\right)} = 1.113$ X. 56

$= \frac{1}{C}\cos^n\theta$ where $C = \frac{4}{\pi}\int_0^{\pi/2}\cos^{n+1}\theta\, d\theta = \frac{4}{\pi}\cdot\frac{2}{\sqrt{\pi}}\cdot\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}+\frac{3}{2}\right)}$

where n is any value X. 63

Series No.

(665) $\sin^4\theta + \frac{4}{3}\cdot\frac{2}{5}\left(1 + \frac{1}{2^2}\right)\sin^6\theta + \dots$

$$+ \frac{4\cdot 6\dots(2n-2)}{5\cdot 7\dots(2n-1)}\cdot\frac{2}{n}\left\{1 + \frac{1}{2^2} + \dots + \frac{1}{(n-1)^2}\right\}\sin^{2n}\theta + \dots\infty$$

(666) $1 + \frac{2^2}{3\cdot 4}\sin^2\theta + \frac{2^2\cdot 4^2}{3\cdot 4\cdot 5\cdot 6}\sin^4\theta$

$$+ \frac{2^2\cdot 4^2\cdot 6^2}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}\sin^6\theta + \dots\ \infty$$

(667) $\frac{1}{2}\cos^2\theta - \frac{1}{3}\left(1 + \frac{1}{2}\right)\cos^3\theta$

$$+ \frac{1}{4}\left(1 + \frac{1}{2} + \frac{1}{3}\right)\cos^4\theta - \dots\ \infty$$

(667a) $\frac{1}{2}\cos 2\theta - \frac{1}{3}\left(1 + \frac{1}{2}\right)\cos 3\theta + \frac{1}{4}\left(1 + \frac{1}{2} + \frac{1}{3}\right)\cos 4\theta - \dots\infty$

(668) $\frac{\sin\theta}{2!} + 1^2\cdot 3^2\left(\frac{1}{1^2} + \frac{1}{3^2}\right)\frac{\sin^3\theta}{4!}$

$$+ 1^2\cdot 3^2\cdot 5^2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\right)\frac{\sin^5\theta}{6!} + \dots\infty$$

(669) $\frac{2}{\pi}\sum_{1}^{\infty}\frac{n}{k^2 + n^2}(1 - \epsilon^{k\pi}\cos n\pi)\sin n\theta$

(670) $\frac{\epsilon^{k\pi} - 1}{k\pi} + \frac{2k}{\pi}\sum_{1}^{\infty}\frac{\epsilon^{k\pi}\cos n\pi - 1}{k^2 + n^2}\cos n\theta$

(671) $\sum_{r=0}^{r=n-1}\frac{x - a\cos\frac{2r\pi + \theta}{n}}{x^2 - 2ax\cos\frac{2r\pi + \theta}{n} + a^2}$

(672) $\tan^{-1}x + (h\sin\theta)\sin\theta - \frac{(h\sin\theta)^2}{2!}\sin 2\theta + \dots\infty$

(673) $1 + \frac{\tan^2\theta}{3} - \frac{2}{3}\frac{\tan^4\theta}{5} + \frac{2\cdot 4}{3\cdot 5}\frac{\tan^6\theta}{7} + \dots\ \infty$

Reference

$= \theta^4$ C. 335

$= \left(\dfrac{\theta}{\sin\theta}\right)^2$ L. 79

$= \dfrac{1}{2}\left[\text{logh } 2\cos^2\dfrac{\theta}{2}\right]^2 \quad \text{where } -\pi < \theta < \pi$ C. 252

$= \dfrac{1}{8}\left[\text{logh } 4\cos^2\dfrac{\theta}{2}\right]^2 - \dfrac{\theta^2}{8} \quad \text{where } -\pi < \theta < \pi$ A. 529

$= \dfrac{\theta^2}{\sin 2\theta}$ L. 82

$= \epsilon^{k\theta} \quad \text{where } 0 < \theta < \pi$

$= \epsilon^{k\theta} \quad \text{where } 0 < \theta < \pi$

$= \dfrac{nx^{n-1}(x^n - a^n\cos\theta)}{x^{2n} - 2x^n a^n \cos\theta + a^{2n}}$ Y. 55

$= \tan^{-1}(x + h) \quad \text{where } x = \cot\theta$ Y. 108

$= \dfrac{gd^{-1}\theta}{\sin\theta}$ Y. 505

Series No.

(674) $1 - \frac{n(n-1)}{2!}\tan^2\theta$

$$+ \frac{n(n-1)(n-2)(n-3)}{4!}\tan^4\theta + \ldots \infty$$

(675) $\sum_{-\infty}^{+\infty}\left\{\tan^{-1}\frac{y}{n+x} - \frac{y}{n}\right\}$

(676) $\sum_{-\infty}^{+\infty}(-1)^n\left\{\tan^{-1}\frac{y}{n+x} - \frac{y}{n}\right\}$

(677) $\cos\theta\cos a + \frac{1}{2}\cos 2\theta\cos 2a + \frac{1}{3}\cos 3\theta\cos 3a + \ldots \infty$

(678) $\cos\theta\cos a - \frac{1}{2}\cos 2\theta\cos 2a + \frac{1}{3}\cos 3\theta\cos 3a - \ldots \infty$

(679) $\sin\theta\sin a + \frac{1}{2}\sin 2\theta\sin 2a + \frac{1}{3}\sin 3\theta\sin 2a + \ldots \infty$

(680) $\sin\theta\sin a - \frac{1}{2}\sin 2\theta\sin 2a + \frac{1}{3}\sin 3\theta\sin 3a - \ldots \infty$

(681) $\cos\theta\cos a + \frac{1}{2^2}\cos 2\theta\cos 2a + \frac{1}{3^2}\cos 3\theta\cos 3a + \ldots \infty$

(682) $\cos\theta\cos a - \frac{1}{2^2}\cos 2\theta\cos 2a + \frac{1}{3^2}\cos 3\theta\cos 3a - \ldots \infty$

(683) $\sin\theta\sin a + \frac{1}{2^2}\sin 2\theta\sin 2a + \frac{1}{3^2}\sin 3\theta\sin 3a + \ldots \infty$

Reference

$= \dfrac{\cos n\theta}{\cos^n \theta}$ where n is a positive integer D. 330

$= -\tan^{-1}\dfrac{y}{x} + \tan^{-1}\dfrac{\tanh \pi y}{\tan \pi x}$ omit $y = x$ A. 314

$= -\tan^{-1}\dfrac{y}{x} + \tan^{-1}\dfrac{\sinh \pi y}{\sin \pi x}$ omit $y = x$ A. 314

$= -\dfrac{1}{4}\operatorname{logh}[4(\cos\theta - \cos a)^2]$ where $\theta \neq 2n\pi + a$ A. 358

$= \dfrac{1}{4}\operatorname{logh}[4(\cos\theta + \cos a)^2]$ where $\theta \neq (2n+1)\pi \pm a$ A. 358

$= \dfrac{1}{4}\operatorname{logh}\dfrac{\sin^2 \frac{1}{2}(\theta + a)}{\sin^2 \frac{1}{2}(\theta - a)}$ where $\theta \neq 2n\pi \pm a$ A. 358

$= \dfrac{1}{4}\operatorname{logh}\dfrac{\cos^2 \frac{1}{2}(\theta - a)}{\cos^2 \frac{1}{2}(\theta + a)}$ where $\theta \neq (2n+1)\pi \pm a$ A. 358

$= \dfrac{1}{4}\theta^2 + \dfrac{1}{4}(a - \pi)^2 - \dfrac{1}{12}\pi^2$ where $0 \leqslant \theta \leqslant a$ A. 361

$= \dfrac{1}{4}a^2 + \dfrac{1}{4}(\theta - \pi)^2 - \dfrac{1}{12}\pi^2$ where $a \leqslant \theta \leqslant \pi$ A. 361

$= \dfrac{1}{12}\pi^2 - \dfrac{1}{4}(a^2 + \theta^2)$ where $-(\pi - a) \leqslant \theta \leqslant (\pi - a)$ A. 361

$= \dfrac{1}{12}\pi^2 - \dfrac{1}{4}\{(a - \pi)^2 + (\theta - \pi)^2\}$ where $(\pi - a) \leqslant \theta \leqslant (\pi + a)$
A. 361

$= \dfrac{1}{2}\theta(\pi - a)$ where $-a \leqslant \theta \leqslant a$ A. 362

$= \dfrac{1}{2}a(\pi - \theta)$ where $a \leqslant \theta \leqslant (2\pi - a)$ A. 362

Series No.

(684) $\sin\theta\cos a + \frac{1}{2}\sin 2\theta\cos 2a + \frac{1}{3}\sin 3\theta\cos 3a + \ldots \infty$

$\cos\theta\sin a + \frac{1}{2}\cos 2\theta\sin 2a + \frac{1}{3}\cos 3\theta\sin 3a + \ldots \infty$

(685) $\cos\theta + a\cos(\theta+\beta) + a^2\cos(\theta+2\beta) + \ldots \infty$

(686) $\sin\theta + a\sin(\theta+\beta) + a^2\sin(\theta+2\beta) + \ldots \infty$

(687) $\cos(\theta+a) + \frac{1}{2}\cos 2(\theta+a) + \frac{1}{3}\cos 3(\theta+a) + \ldots \infty$

$\theta \neq 2n\pi \pm a$

(688) $\cos(\theta-a) + \frac{1}{2}\cos 2(\theta-a) + \frac{1}{3}\cos 3(\theta-a) + \ldots \infty$

$\theta \neq 2n\pi \pm a$

(689) $\sin(\theta+a) + \frac{1}{2}\sin 2(\theta+a) + \frac{1}{3}\sin 3(\theta+a) + \ldots \infty$

(690) $\sin(\theta-a) + \frac{1}{2}\sin 2(\theta-a) + \frac{1}{3}\sin 3(\theta-a) - \ldots \infty$

(691) $\sin\theta + a\sin(\theta+\beta) + \frac{a^2}{2!}\sin(\theta+2\beta) + \ldots \infty$

Reference

$$= f(\theta) \quad \text{where } f(\theta) = -\frac{1}{2}\theta, \quad g(\theta) = \frac{1}{2}(\pi - a) \quad \text{if } 0 < \theta < a$$

$$= g(\theta) \quad \text{where } f(\theta) = \frac{1}{2}(\pi - \theta), \quad g(\theta) = -\frac{1}{2}a \quad \text{if } a < \theta < \pi$$

$$f(a) = \frac{1}{4}(\pi - 2a) = g(a)$$ A. 358

$$= \frac{\cos\theta - a\cos(\theta - \beta)}{1 - 2a\cos\beta + a^2}$$ E. 131

$$= \frac{\sin\theta - a\sin(\theta - \beta)}{1 - 2a\cos\beta + a^2}$$ E. 117

$$= -\frac{1}{2}\,\text{logh}\left[4\sin^2\frac{1}{2}(\theta + a)\right]$$ A. 358

$$= -\frac{1}{2}\,\text{logh}\left[4\sin^2\frac{1}{2}(\theta - a)\right]$$ A. 358

$= \frac{1}{2}\{\pi - (\theta + a)\}$ where $0 < a < \pi$ and $0 < (\theta + a) < 2\pi$

A. 358

$= \frac{1}{2}\{\pi - (\theta - a)\}$ where $0 < a < \pi$ and $0 < (\theta - a) < 2\pi$;

when $\theta < a$ the sum is diminished by π A. 358

$= \epsilon^{a\cos\beta}\sin(\theta + a\sin\beta)$ E. 121

Series No.

(692) $\cos\theta + a\cos(\theta+\beta) + \frac{a^2}{2!}\cos(\theta+2\beta) + \ldots \infty$

(693) $1 - \cos\theta\cos\beta + \frac{\cos^2\theta}{2!}\cos 2\beta - \frac{\cos^3\theta}{3!}\cos 3\beta + \ldots \infty$

(694) $\sin\theta - \frac{1}{2!}\sin(\theta+2\beta) + \frac{1}{4!}\sin(\theta+4\beta) - \ldots \infty$

(695) $\cos\theta - \frac{1}{3!}\cos(\theta+2\beta) + \frac{1}{5!}\cos(\theta+4\beta) - \ldots \infty$

(696) $a\cos\theta - \frac{a^3}{3}\cos(\theta+2\beta) + \frac{a^5}{5}\cos(\theta+4\beta) + \ldots \infty$

(697) $\sum_{1}^{\infty} \frac{1}{n^2}\sin n\theta \sin na \cos n\beta$

(698) $\sin\theta\cos\theta + \sin^2\theta\cos 2\theta + \ldots \infty$

(699) $\sum_{1}^{\infty} \frac{1}{n}\sin 2n\theta \sin^2 n\phi$

(700) $\sum_{1}^{\infty} \frac{1}{n^2}\sin^2 n\theta \sin^2 n\phi$

Reference

$= \epsilon^{a \cos\beta} \cos(\theta + a \sin\beta)$ E. 121

$= \epsilon^{-\cos\theta\cos\beta} \cos(\cos\theta \sin\beta)$ E. 121

$\doteq \sin\theta \cos(\cos\beta) \cosh(\sin\beta) - \cos\theta \sin(\cos\beta) \sinh(\sin\beta)$ E. 121

$= \sin(\cos\beta) \cosh(\sin\beta) \cos(\theta - \beta) - \cos(\cos\beta) \sinh(\sin\beta) \sin(\theta - \beta)$ E. 122

$= \frac{1}{2} \cos(\theta - \beta) \tan^{-1} \frac{2a\cos\beta}{1 - a^2} - \frac{1}{2} \sin(\theta - \beta) \tanh^{-1} \frac{2a\sin\beta}{1 + a^2}$ E. 122

where $0 < \beta < a < \frac{\pi}{2}$

$= \frac{1}{2} \theta(\pi - a)$ where $0 < \theta < (a - \beta)$

$= \frac{1}{4} \pi(\theta + a - \beta) - \frac{1}{2} a\theta$ where $(a - \beta) < \theta < (a + \beta)$

$= \frac{1}{2} a(\pi - \theta)$ where $(a + \beta) < \theta < \pi$ A. 390

$= \frac{\sin\theta(\cos\theta - \sin\theta)}{1 - \sin 2\theta + \sin^2\theta}$ where $\theta \neq \pm \frac{\pi}{2}$ E. 117

$= \frac{1}{4} \pi$ where $\begin{matrix} 0 < 2\theta < \pi \\ \theta < \phi < (\pi - \theta) \end{matrix}$ A. 391

$= \frac{1}{4} \pi\theta$ A. 391

Series No.

(701) $\sum_{1}^{\infty} \frac{1}{n^2} \sin^4 n\theta \sin^2 n\phi$

(702) $\sum_{1}^{\infty} \frac{1}{n^4} \sin^4 n\theta \sin^2 n\phi$

(703) $\cos\theta \sin\theta + \frac{1}{2}\cos^2\theta \sin 2\theta + \frac{1}{3}\cos^3\theta \sin 3\theta + \ldots \infty$

(704) $\sec^2\theta - \frac{1}{3}\tan^2\theta \sec^2\theta + \frac{1}{5}\tan^4\theta \sec^2\theta + \ldots \infty$

(705) $\cot\theta \operatorname{cosec}^2\theta - \cot 3\theta \operatorname{cosec}^2 3\theta + \cot 5\theta \operatorname{cosec}^2 5\theta + \ldots n$ terms

(706) $\theta + \cos\theta \sin\theta + \frac{\cos^2\theta}{2}\sin 2\theta + \frac{\cos^3\theta}{3}\sin 3\theta + \ldots \infty$

(707) $\frac{\sin\theta}{\cos\theta} + \frac{1}{2}\frac{\sin 2\theta}{\cos^2\theta} + \frac{1}{3}\frac{\sin 3\theta}{\cos^3\theta} + \ldots \infty$

(708) $\operatorname{logh}\sin\theta + a\cot\theta - \frac{a^2}{2}\operatorname{cosec}^2\theta + \frac{a^3}{3}\frac{\cos\theta}{\sin^3\theta} - \ldots \infty$

(709) $\frac{1}{n} + \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{1}{m}\tan\frac{m\pi}{n}\cos 2m\pi\theta$

(710) $\frac{1}{3} + 4\pi^{-1}\sum_{m=1}^{\infty} m^{-1}\sin\frac{2}{3}m\pi \cos 2m\pi\theta$

Reference

$= \frac{1}{8} \pi\theta$ A. 391

$= \frac{1}{6} \pi\theta^3$ A. 391

$= \frac{\pi}{2} - \theta \quad \text{where } 0 < \theta < \frac{\pi}{2}$ E. 121

$= \frac{\theta}{\sin\theta\cos\theta}$ 2Z. 246

$= 2n^3$ A. 223

$= \frac{\pi}{2}$ Y. 108

$= \frac{\pi}{2}$ L. 81

$= \text{logh} \sin(\theta + a)$ L. 71

$= (-1)^s \quad \text{if } \theta \text{ is not a multiple of } \frac{1}{n}$

$= 0 \quad \text{if } \theta \text{ is a multiple integer of } \frac{1}{n}$ Q. 191

$$\begin{cases} \theta \text{ is a real variable between 0 and 1} \\ n \text{ is an odd integer} \geqslant 3 \\ s \text{ is the greatest integer in } n\theta \end{cases}$$

$= 1 \quad \text{when } 0 < \theta < \frac{1}{3} \text{ and } \frac{2}{3} < \theta < 1$

$= -1 \quad \text{when } \frac{1}{3} < \theta < \frac{2}{3}$ Q. 191

Series No.

XV. Hyperbolic Summations

(711) $1 + \cosh\theta + \dfrac{\cosh 2\theta}{2!} + \dfrac{\cosh 3\theta}{3!} + \dots \infty$

(712) $\sinh\theta + \dfrac{\sinh 2\theta}{2!} + \dfrac{\sinh 3\theta}{3!} + \dots \infty$

(713) $1 - \dfrac{2}{3}\sinh^2\theta + \dfrac{2\cdot 4}{3\cdot 5}\sinh^4\theta - \dots \infty$

(714) $1 + a\cosh\theta + a^2\cosh 2\theta + \dots a^{n-1}\cosh(n-1)\theta$

(715) $a\sinh\theta + a^2\sinh 2\theta + \dots \infty$

(716) $\sinh\theta - \dfrac{1}{2}\sinh 2\theta + \dfrac{1}{3}\sinh 3\theta - \dots \infty$

(717) $\sinh^2\theta - 2^2\sinh^4\dfrac{\theta}{2} - 2^4\sinh^4\dfrac{\theta}{2^2} - 2^6\sinh^4\dfrac{\theta}{2^3} - \dots \infty$

(718) $\tanh\theta + \dfrac{1}{3}\tanh^3\theta + \dfrac{1}{5}\tanh^5\theta + \dots \infty$

(719) $\coth^2\theta - \sum_{1}^{n}\dfrac{1}{2^{2n}}\tanh^2\dfrac{\theta}{2^n}$

(720)† $\dfrac{1}{n}\sum_{r=0}^{r=\frac{n}{2}-1}\dfrac{\tanh\theta\dfrac{2}{n\sin^2\dfrac{2r+1}{2n}\pi}}{1+\dfrac{\tanh^2\theta}{\tan^2\dfrac{2r+1}{2n}\pi}}$

† $\dfrac{\dfrac{2\tanh\theta}{n\sin^2 R\pi}}{1+\dfrac{\tanh^2\theta}{\tan^2 R\pi}} = \dfrac{2\sinh 2\theta}{n\{\cosh 2\theta - \cos 2R\pi\}}$

Reference

$= \epsilon^{\cosh\theta} \cosh(\sinh\theta)$ E. 122

$= \epsilon^{\cosh\theta} \sinh(\sinh\theta)$ E. 122

$= \dfrac{\theta}{\sinh\theta \cosh\theta}$ A. 198

$= \dfrac{1 - a\cosh\theta - a^n \cosh n\theta + a^{n+1}\cosh(n-1)\theta}{1 - 2a\cosh\theta + a^2}$ E. 117

$= \dfrac{a \sinh\theta}{1 - 2a\cosh\theta + a^2}$ E. 117

$= \dfrac{\theta}{2}$ E. 122

$= \theta^2$ C. 336

$= \tan\theta - \dfrac{1}{3}\tan^3\theta + \dfrac{1}{5}\tan^5\theta - \ldots \infty$ where θ is between $\pm\dfrac{\pi}{4}$ E. 123

$= \dfrac{1}{2^{2n}} \coth^2 \dfrac{\theta}{2^{2n}}$ C. 315

$= \tanh n\theta$ where n is even

Series No.

(721)† $$\frac{1}{n}(\tanh\theta + \coth\theta) + \sum_{r=1}^{r=\frac{n}{2}-1} \frac{\tanh\theta \dfrac{2}{n\sin^2\dfrac{r\pi}{n}}}{1 + \dfrac{\tanh^2\theta}{\tan^2\dfrac{r\pi}{n}}}$$

(722)† $$\frac{\tanh\theta}{n} + \sum_{r=0}^{r=\frac{n-3}{2}} \frac{\tanh\theta \dfrac{2}{n\sin^2\dfrac{2r+1}{2n}\pi}}{1 + \dfrac{\tanh^2\theta}{\tan^2\dfrac{2r+1}{2n}\pi}}$$

(723)† $$\frac{\coth\theta}{n} + \sum_{r=1}^{r=\frac{n-1}{2}} \frac{\tanh\theta \dfrac{2}{n\sin^2\dfrac{r\pi}{n}}}{1 + \dfrac{\tanh^2\theta}{\tan^2\dfrac{r\pi}{n}}}$$

(724)‡ $$\frac{1}{n} \sum_{r=0}^{r=\frac{n}{2}-1} \frac{1}{\dfrac{\sin^2\dfrac{2r+1}{2n}\pi}{\sinh\theta} + \dfrac{1}{2}\tanh\dfrac{\theta}{2}}$$

† $$\frac{\dfrac{2\tanh\theta}{n\sin^2 R\pi}}{1 + \dfrac{\tanh^2\theta}{\tan^2 R\pi}} = \frac{2\sinh 2\theta}{n\{\cosh 2\theta - \cos 2R\pi\}}$$

‡ $$\frac{1}{\dfrac{\sin^2 R\pi}{\sinh\theta} + \dfrac{1}{2}\tanh\dfrac{\theta}{2}} = \frac{2\sinh\theta}{\cosh\theta - \cos 2R\pi}$$

$= \coth n\theta$ where n is even

$= \tanh n\theta$ where n is odd

$= \coth n\theta$ where n is odd

$= \tanh \frac{n\theta}{2}$ where n is even

Series No.

(725)‡ $$\frac{1}{n}\left\{2\coth\theta + \sum_{r=1}^{r=\frac{n}{2}-1} \frac{1}{\dfrac{\sin^2 \frac{r\pi}{n}}{\sinh\theta} + \frac{1}{2}\tanh\frac{\theta}{2}}\right\}$$

(726)‡ $$\frac{1}{n}\left\{\tanh\frac{\theta}{2} + \sum_{r=0}^{r=\frac{n-3}{2}} \frac{1}{\dfrac{\sin^2 \frac{2r+1}{2n}\pi}{\sinh\theta} + \frac{1}{2}\tanh\frac{\theta}{2}}\right\}$$

(727)‡ $$\frac{1}{n}\left\{\coth\frac{\theta}{2} + \sum_{r=1}^{r=\frac{n-1}{2}} \frac{1}{\dfrac{\sin^2 \frac{r\pi}{n}}{\sinh\theta} + \frac{1}{2}\tanh\frac{\theta}{2}}\right\}$$

(728) $$\frac{1}{a} + 2a\sum_{1}^{\infty}\frac{\cos n\theta}{a^2+n^2}$$

(729) $$2\sum_{1}^{\infty}\frac{n\sin n\theta}{n^2+a^2}$$

(730) $$\frac{1}{a} + 2a\sum_{1}^{\infty}\frac{(-1)^n\cos n\theta}{a^2+n^2}$$

(731) $$2\sum_{1}^{\infty}\frac{(-1)^{n-1}n\sin n\theta}{n^2+a^2}$$

XVI. Trigonometrical Expansions

(732) $$1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots \infty = \sum_{1}^{\infty}(-1)^n\frac{\theta^{2n}}{(2n)!}$$

(733) $$\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots \infty = \sum_{0}^{\infty}(-1)^n\frac{\theta^{2n+1}}{(2n+1)!}$$

‡ $$\frac{1}{\dfrac{\sin^2 R\pi}{\sinh\theta} + \frac{1}{2}\tanh\frac{\theta}{2}} = \frac{2\sinh\theta}{\cosh\theta - \cos 2R\pi}$$

Reference

$= \coth \dfrac{n\theta}{2}$ where n is even

$= \tanh \dfrac{n\theta}{2}$ where n is odd

$= \coth \dfrac{n\theta}{2}$ where n is odd

$= \pi \dfrac{\cosh a(\pi - \theta)}{\sinh a\pi}$ where $0 \leqslant \theta \leqslant 2\pi$ A. 393

$= \pi \dfrac{\sinh (\pi - \theta)a}{\sinh a\pi}$ where $0 < \theta < 2\pi$ A. 393

$= \pi \dfrac{\cosh a\theta}{\sinh a\pi}$ where $-\pi \leqslant \theta \leqslant \pi$ A. 368

$= \pi \dfrac{\sinh a\theta}{\sinh a\theta}$ where $-\pi < \theta < \pi$ A. 368

$= \cos \theta$ where $\theta < \infty$

$= \sin \theta$ where $\theta < \infty$

Series No.

(734) $\theta - \frac{\theta^3}{3} + \frac{\theta^5}{5} - \ldots \infty = \sum_0^\infty (-1)^n \frac{\theta^{2n+1}}{2n+1}$

In the following series Nos. (735) through (763), see No. (1130) for values of the coefficients.

(735) $1 - B_1 \frac{\theta^2}{2!} - B_2 \frac{\theta^4}{4!} - \ldots \infty = 1 - \sum_1^\infty B_n \frac{\theta^{2n}}{(2n)!}$

(736) $\alpha_1 \frac{\theta}{2!} + \alpha_2 \frac{\theta^3}{4!} + \ldots \infty = \sum_1^\infty \alpha^n \frac{\theta^{2n-1}}{(2n)!}$

(737) $1 + \beta_1 \frac{\theta^2}{2!} + \beta_2 \frac{\theta^4}{4!} + \ldots \infty = 1 + \sum_1^\infty \beta_n \frac{\theta^{2n}}{(2n)!}$

(738) $1 + E_2 \frac{\theta^2}{2!} + E_4 \frac{\theta^4}{4!} + \ldots \infty = 1 + \sum_1^\infty E_{2n} \frac{\theta^{2n}}{(2n)!}$

(739) $2 + \delta_1 \frac{\theta^2}{2!} + \delta_2 \frac{\theta^4}{4!} + \ldots \infty = 2 + \sum_1^\infty \delta_n \frac{\theta^{2n}}{(2n)!}$

(740) $\zeta_1 \frac{\theta}{2!} + \zeta_2 \frac{\theta^3}{4!} + \zeta_3 \frac{\theta^5}{6!} + \ldots \infty = \sum_1^\infty \zeta_n \frac{\theta^{2n-1}}{(2n)!}$

(741) $I_0 + I_1 \frac{\theta^2}{2!} + I_2 \frac{\theta^4}{4!} + \ldots \infty = I_0 + \sum_1^\infty I_n \frac{\theta^{2n}}{(2n)!}$

(742) $H_0 + H_1 \frac{\theta^2}{2!} + H_2 \frac{\theta^4}{4!} + \ldots \infty = H_0 + \sum_1^\infty H_n \frac{\theta^{2n}}{(2n)!}$

(743) $\gamma_1 \frac{\theta}{2!} + \gamma_2 \frac{\theta^3}{4!} + \gamma_3 \frac{\theta^5}{6!} + \ldots \infty = \sum_1^\infty \gamma_n \frac{\theta^{2n-1}}{(2n)!}$

Reference

$= \tan^{-1}\theta \quad \text{where } \theta^2 \leqslant 1$ E. 107

$= \frac{\theta}{2}\cot\frac{\theta}{2}$ AC. 41

$= \tan\frac{\theta}{2}$ AC. 41

$= \theta \operatorname{cosec}\theta$ AC. 41

$= \sec\theta$ AC. 41

$= \dfrac{3\theta\cos\frac{\theta}{2}}{\sin\frac{3}{2}\theta}$ AC. 41

$= \dfrac{\sin\frac{\theta}{2}}{\cos\frac{3\theta}{2}}$ AC. 41

$= \dfrac{3\sin\frac{\theta}{2}}{2\sin\frac{3\theta}{2}}$ AC. 41

$= \dfrac{3\cos\frac{\theta}{2}}{2\cos\frac{3\theta}{2}}$ AC. 41

$= \dfrac{3\sin\frac{\theta}{2}\sin\theta}{2\sin\frac{3\theta}{2}}$ AC. 41

Series No.

(744) $\theta_1 \frac{\theta}{2!} + \theta_2 \frac{\theta^3}{4!} + \theta_3 \frac{\theta^5}{6!} + \ldots \infty = \sum_1^\infty \theta_n \frac{\theta^{2n-1}}{(2n)!}$

(745) $1 - \epsilon_1 \frac{\theta^2}{2!} - \epsilon_2 \frac{\theta^4}{4!} - \ldots \infty = 1 - \sum_1^\infty \epsilon_n \frac{\theta^{2n}}{(2n)!}$

(746) $T_1\theta + T_2 \frac{\theta^3}{3!} + T_3 \frac{\theta^5}{5!} + \ldots \infty = \sum_1^\infty T_n \frac{\theta^{2n-1}}{(2n-1)!}$

(747) $J_0 + J_1 \frac{\theta^2}{2!} + J_2 \frac{\theta^4}{4!} + \ldots \infty = J_0 + \sum_1^\infty J_n \frac{\theta^{2n}}{(2n)!}$

(748) $S_0 + S_1 \frac{\theta^2}{2!} + S_2 \frac{\theta^4}{4!} + \ldots \infty = S_0 + \sum_1^\infty S_n \frac{\theta^{2n}}{(2n)!}$

(749) $\eta_1 \frac{\theta}{2!} + \eta_2 \frac{\theta^3}{4!} + \eta_3 \frac{\theta^5}{6!} + \ldots \infty = \sum_1^\infty \eta_n \frac{\theta^{2n-1}}{(2n)!}$

(750) $R_0 + R_1 \frac{\theta^2}{2!} + R_2 \frac{\theta^4}{4!} + \ldots \infty = R_0 + \sum_1^\infty R_n \frac{\theta^{2n}}{(2n)!}$

(751) $P_0 + P_1 \frac{\theta^2}{2!} + P_2 \frac{\theta^4}{4!} + \ldots \infty = P_0 + \sum_1^\infty P_n \frac{\theta^{2n}}{(2n)!}$

(752) $Q_1\theta + Q_2 \frac{\theta^3}{2!} + Q_3 \frac{\theta^5}{4!} + \ldots \infty = \sum_1^\infty Q_n \frac{\theta^{2n-1}}{(2n-2)!}$

(753) $1 + E_1\theta + E_2 \frac{\theta^2}{2!} + E_3 \frac{\theta^3}{3!} + \ldots \infty$

(754) $\theta + E_2 \frac{\theta^3}{3!} + E_4 \frac{\theta^5}{5!} + \ldots \infty$

(755) $\operatorname{logh} \theta + \sum_1^\infty \frac{(2^{2n-1} - 1)2^{2n}}{\mathrm{n} \cdot (2n)!} B_n\theta^{2n}$

(756) $B_1\theta^2 + B_2 \frac{2^3}{2 \cdot 4!} \theta^4 + B_3 \frac{2^5}{3 \cdot 6!} \theta^6 + \ldots \infty$

	Reference
$= \dfrac{\sin\theta\cos\frac{\theta}{2}}{2\cos\frac{3\theta}{2}}$	AC. 41
$= \dfrac{3\theta\cos 2\theta}{\sin 3\theta}$	AC. 41
$= \dfrac{\sin 2\theta}{2\cos 3\theta}$	AC. 41
$= \dfrac{3\sin 2\theta}{\sin 3\theta}$	AC. 41
$= \dfrac{\cos 2\theta}{\cos 3\theta}$	AC. 41
$= \dfrac{3\sin^2\theta}{2\sin 3\theta}$	AC. 41
$= \dfrac{\cos^2\theta}{\cos 3\theta}$	AC. 41
$= \dfrac{\cos\theta}{\cos 2\theta}$	AC. 41
$= \dfrac{\sin\theta}{\cos 2\theta}$	AC. 41
$= \sec\theta + \tan\theta$	Y. 500
$= \text{logh}\,(\sec\theta + \tan\theta) = gd^{-1}\theta$	Y. 500
$= \text{logh}\tan\theta \quad \text{where } \theta^2 < \dfrac{\pi^2}{4}$	T. 123
$= -\,\text{logh}\dfrac{\sin\theta}{\theta}$	B. 245

Series No.

(757) $\sum_{1}^{\infty} \text{logh}\left\{1 - \frac{4\theta^2}{(2n-1)^2\pi^2}\right\} = -\sum_{1}^{\infty} \frac{2^{2n-1}(2^n - 1)}{n(2n)!} B_n\theta^{2n}$

(758) $\text{logh}\,\theta + \sum_{1}^{\infty} \text{logh}\left\{1 - \frac{\theta^2}{n^2\pi^2}\right\} = \text{logh}\,\theta - \sum_{1}^{\infty} \frac{2^{2n-1}}{n(2n)!} B_n\theta^{2n}$

(759) $\frac{B_1}{2}\cdot\frac{\theta^2}{2!} + \frac{B_2}{4}\cdot\frac{\theta^4}{4!} + \frac{B_3}{6}\cdot\frac{\theta^6}{6!} + \dots \infty$

(760) $-\sum_{1}^{\infty} 2^{n+1} \cos\frac{n\pi}{2}\frac{B_n}{2!}\frac{\theta^{2n}}{(2n)!}$

(761) $\frac{1}{\theta} + \frac{\theta}{3!} + \frac{7}{3\cdot 5!}\theta^3 + \dots \infty$

$$= \frac{1}{\theta} + \sum_{0}^{\infty} \frac{2(2^{2n+1} - 1)}{(2n+2)!} B_{n+1}\theta^{2n+1}$$

(762) $\frac{1}{\theta} - \frac{\theta}{3} - \frac{\theta^3}{45} - \dots \infty = \frac{1}{\theta} - \sum_{1}^{\infty} \frac{2^{2n}}{(2n)!} B_n\theta^{2n-1}$

(763) $\theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \dots \infty = \sum_{1}^{\infty} \frac{2^{2n}(2^{2n} - 1)}{(2n)!} B_n\theta^{2n-1}$

(764) $\sum_{0}^{\infty} \frac{8\theta}{(2n+1)^2\pi^2 - 4\theta^2}\,\theta$

(765) $\frac{2}{\pi - 2\theta} - \frac{2}{\pi + 2\theta} + \frac{2}{3\pi - 2\theta} - \frac{2}{3\pi + 2\theta} + \dots \infty$

(766) $2\sum_{1}^{\infty} \frac{\theta}{\left(\frac{2n-1}{2}\right)^2\pi^2 - \theta^2}$

(767) $\frac{1}{\theta} + \frac{1}{\theta - \pi} + \frac{1}{\theta + \pi} + \frac{1}{\theta - 2\pi} + \frac{1}{\theta + 2\pi} + \dots \infty$

$$= \frac{1}{\theta} - \sum_{1}^{\infty} \frac{2\theta}{n^2\pi^2 - \theta^2}$$

(768) $\frac{4}{\pi - 2\theta} - \frac{4}{3\pi + 2\theta} + \frac{4}{5\pi - 2\theta} - \frac{4}{7\pi + 2\theta} + \dots \infty$

Reference

$= \text{logh} \cos\theta \quad \text{where } \theta^2 < \frac{\pi^2}{4}$ B. 237

$= \text{logh} \sin\theta \quad \text{where } \theta^2 < \pi^2$ B. 237

$= \text{logh} \dfrac{\theta}{2 \sin \frac{\theta}{2}}$ A. 315

$= \text{logh} \dfrac{\cosh\theta - \cos\theta}{\theta^2}$ A. 315

$= \operatorname{cosec}\theta \quad \text{where } \theta^2 < \pi^2$ T. 121

$= \cot\theta \quad \text{where } \theta^2 < \pi^2$ T. 121

$= \tan\theta \quad \text{where } \theta^2 < \frac{\pi^2}{4}$ T. 121

$= \tan\theta \quad \text{where } \theta \neq n\pi$ A. 296

$= \tan\theta \quad \text{where } \theta \neq \frac{(2n+1)\pi}{2}$ C. 360

$= \tan\theta$ B. 237

$= \cot\theta \quad \text{where } \theta \neq n\pi$ C. 360

$= \sec\theta + \tan\theta \quad \text{where } \frac{\pi}{2} > \theta > -\frac{\pi}{2}$ Y. 501

Series No.

(769) $$\frac{1}{\theta}+\frac{1}{\pi-\theta}-\frac{1}{\pi+\theta}-\frac{1}{2\pi-\theta}+\frac{1}{2\pi+\theta}+\dots\infty$$
$$=\frac{1}{\theta}+2\theta\sum_{1}^{\infty}\frac{(-1)^{n-1}}{n^2\pi^2-\theta^2}$$

(770) $$\frac{2}{\pi-2\theta}+\frac{2}{\pi+2\theta}-\frac{2}{3\pi-2\theta}-\frac{2}{3\pi+2\theta}-\dots\infty$$
$$=\sum_{1}^{\infty}(-1)^{n-1}\frac{(2n-1)\pi}{\left(\frac{2n-1}{2}\right)^2\pi^2-\theta^2}$$

(771) $$\frac{1}{\pi-4\theta^2}-\frac{3}{3^2\pi^2-4\theta^2}+\frac{5}{5^2\pi^2-4\theta^2}-\dots\infty$$

(772) $$\frac{1}{\theta^2}-\frac{1}{\theta^2-\pi^2}+\frac{1}{\theta^2-2^2\pi^2}-\dots\infty$$

(773) $$\frac{1}{(\pi-2\theta)^2}+\frac{1}{(\pi+2\theta)^2}+\frac{1}{(3\pi-2\theta)^2}+\frac{1}{(3\pi+2\theta)^2}+\dots\infty$$

(774) $$1+2\theta+2\theta^2+\frac{8\theta^3}{3}+\frac{10\theta^4}{3}+\dots\infty$$

(775) $$\theta+\frac{1}{2}\cdot\frac{\theta^3}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{\theta^5}{5}+\dots\infty=\sum_{0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2(2n+1)}\theta^{2n+1}$$

(776) $$1+\frac{\theta^2}{3!}+\frac{14\theta^4}{6!}+\dots\infty$$

(777) $$1-\theta^2+\frac{2^3\theta^4}{4!}-\frac{2^5\theta^6}{6!}+\dots\infty$$

(778) $$\theta^2+\frac{2}{3}\cdot\frac{\theta^4}{2}+\frac{2\cdot 4}{3\cdot 5}\cdot\frac{\theta^6}{3}+\dots\infty=\sum_{0}^{\infty}\frac{2^{2n}(n!)^2}{(2n+1)!(n+1)}\theta^{2n+2}$$

(779) $$1+\frac{1}{3\cdot 4}\theta+\frac{1\cdot 3}{5\cdot 4^2}\frac{\theta^2}{2!}+\dots\infty$$
$$=1+\sum_{1}^{\infty}\frac{1\cdot 3\dots(2n-1)}{(2n+1)4^n}\cdot\frac{\theta^n}{n!}$$

(780) $$\theta+\frac{1}{3}\cdot\frac{\theta^2}{2}+\frac{1\cdot 2}{3\cdot 5}\frac{\theta^3}{3}+\dots\infty=\theta+\sum_{2}^{\infty}\frac{(n-1)!}{3\cdot 5\dots 2n-1}\frac{\theta^n}{n}$$

Reference

$= \operatorname{cosec}\theta \quad \text{where } \theta \neq n\pi$ C. 361

$= \sec\theta \quad \text{where } \theta \neq \frac{2n \pm 1}{2}\pi$ C. 361

$= \frac{\sec\theta}{4\pi}$ E. 158

$= \frac{1 + \theta \operatorname{cosec}\theta}{2\theta^2}$ E. 158

$= \frac{\sec^2\theta}{4}$ E. 159

$= \tan\left\{\frac{\pi}{4} + \theta\right\}$ H. 498

$= \sin^{-1}\theta = \frac{\pi}{2} - \cos^{-1}\theta \quad \text{where } \theta^2 \leqslant 1$ T. 121

$= \frac{\theta}{\sin\theta}$ H. 498

$= \cos^2\theta$ H. 498

$= (\sin^{-1}\theta)^2 \quad \text{where } |\theta^2| < |$ T. 122

$= \frac{\operatorname{vers}^{-1}\theta}{\sqrt{2\theta}}$ Y. 505

$= \frac{(\operatorname{vers}^{-1}\theta)^2}{2}$ Y. 505

Series No.

(781) $\frac{\theta^5}{1\cdot3\cdot5}-\frac{\theta^7}{3\cdot5\cdot7}+\frac{\theta^9}{5\cdot7\cdot9}-\ldots\infty$

(782) $\theta^2-\frac{\theta^4}{3}+\frac{2\theta^6}{45}-\ldots\infty$

(783) $\theta+\frac{2^2}{3!}\theta^3+\frac{2^2.4^2}{5!}\theta^5+\frac{2^2.4^2.6^2}{7!}\theta^7+\ldots\infty$

(784) $\theta-\frac{\theta^3}{3}-\frac{2}{3}\cdot\frac{\theta^5}{5}-\frac{2\cdot4}{3\cdot5}\frac{\theta^7}{7}-\ldots\infty$

(785) $\frac{\theta^3}{1\cdot3}-\frac{\theta^5}{3\cdot5}+\frac{\theta^7}{5\cdot7}-\ldots\infty$

(786) $1+\theta-\frac{2\theta^3}{3!}-\frac{2^2\theta^4}{4!}-\frac{2^2\theta^5}{5!}+\frac{2^3\theta^7}{7!}-\ldots\infty$

(787) $\theta+\theta^2+\frac{2\theta^3}{3!}-\frac{2^2\theta^5}{5!}-\frac{2^3\theta^6}{6!}-\frac{2^3\theta^7}{7!}-\ldots\infty$

(788) $\frac{\theta^2}{3}+\frac{7\theta^4}{90}+\frac{17\theta^6}{30}+\ldots\infty$

(789) $2\theta+\frac{4}{3}\theta^3+\frac{4}{3}\theta^5+\ldots\infty$

(790) $\theta-\frac{\theta^2}{2}+\frac{\theta^3}{6}-\frac{\theta^4}{12}+\frac{\theta^5}{24}-\ldots\infty$

(791) $-\frac{\theta^2}{3}+\frac{13}{90}\theta^4-\frac{251}{5\cdot7\cdot9^2}\theta^6+\ldots\infty$

(793)† $\theta^2-\frac{2}{3}\theta^4+\frac{61}{120}\theta^6-\ldots\infty$

(794) $-\frac{\theta^2}{3}-\frac{7}{90}\theta^4-\frac{62}{2835}\theta^6-\ldots\infty$

(795) $\theta-\frac{\theta^2}{2}+\frac{2\theta^3}{3}-\ldots\infty$

(796) $\frac{\theta^2}{2}+\frac{\theta^4}{12}+\frac{\theta^6}{45}+\ldots\infty$

† No. (792) has been omitted because it duplicates a previous series.—Ed.

	Reference
$= \dfrac{(\theta^2 + 1)^2}{8} \tan^{-1}\theta - \dfrac{5\theta^3 + 3\theta}{24}$	1Z. 135
$= \sin^2\theta$	D. 336
$= \dfrac{\sin^{-1}\theta}{\sqrt{1 - \theta^2}}$ where $\lvert\theta\rvert < 1$	L. 78
$= \sqrt{1 - \theta^2}\,\sin^{-1}\theta$	A. 191
$= \dfrac{1}{2}(1 + \theta^2)\tan^{-1}\theta - \dfrac{1}{2}\theta$	H. 475
$= \epsilon^{\theta}\cos\theta$	H. 497
$= \epsilon^{\theta}\sin\theta$	H. 497
$= \operatorname{logh}\dfrac{\tan\theta}{\theta}$	L. 80
$= \operatorname{logh}\tan\left\{\dfrac{\pi}{4} + \theta\right\}$	H. 498
$= \operatorname{logh}(1 + \sin\theta)$	H. 498
$= \operatorname{logh}\dfrac{\tan^{-1}\theta}{\theta}$	Y. 80
$= \operatorname{logh}(1 + \theta\sin\theta)$	Y. 106
$= \operatorname{logh}\theta\cot\theta$	Y. 80
$= \operatorname{logh}(1 + \tan\theta)$	L. 79
$= \operatorname{logh}\sec\theta$	H. 497

Series No.

(797) $1 + \theta + \frac{\theta^2}{2} - \frac{\theta^4}{8} - \frac{\theta^5}{15} + \frac{\theta^6}{240} - \dots \infty$

(798) $\epsilon\left\{1 - \frac{\theta^2}{2!} + \frac{4\theta^4}{4!} - \frac{31\theta^6}{6!} + \dots \infty\right\}$

(799) $1 + \theta^2 + \frac{1}{3}\theta^4 + \frac{1}{120}\theta^6 + \dots \infty$

(800) $-\frac{\theta^2}{2!} - \frac{2\theta^4}{4!} - \frac{16\theta^6}{6!} - \frac{272\theta^8}{8!} - \dots \infty$

(801) $1 + \theta + \frac{\theta^2}{2!} + \frac{2\theta^3}{3!} + \frac{5\theta^4}{4!} + \dots \infty$

(802) $1 + \theta + \frac{\theta^2}{2} - \frac{\theta^3}{3} - \frac{11\theta^4}{24} - \frac{\theta^5}{5} - \dots \infty$

(803) $1 + \frac{\theta^3}{3!} + \frac{\theta^6}{6!} + \frac{\theta^9}{9!} + \dots \infty$

(804) $\theta + \frac{\theta^4}{4!} + \frac{\theta^7}{7!} + \frac{\theta^{10}}{10!} + \dots \infty$

(805) $\frac{\pi}{2} - \frac{1}{\theta} + \frac{1}{3\theta^3} - \frac{1}{5\theta^5} + \frac{1}{7\theta^7} - \dots \infty$

$$= \frac{\pi}{2} - \sum_0^\infty (-1)^n \frac{1}{(2n+1)\theta^{2n+1}}$$

(806) $\frac{\pi}{2} - \frac{1}{\theta} - \frac{1}{2\cdot 3}\frac{1}{\theta^3} - \dots \infty = \frac{\pi}{2} - \sum_0^\infty \frac{(2n)!}{2^{2n}(n!)^2(2n+1)}\theta^{-2n-1}$

(807) $1 + \theta + \frac{\theta^2}{2!} + \frac{3\theta^3}{3!} + \frac{9\theta^4}{4!} + \frac{37\theta^5}{5!} + \dots \infty$

(808) $1 + \theta + \frac{\theta^2}{2} - \frac{\theta^3}{6} + \frac{7\theta^4}{24} - \dots \infty$

(809) $1 + \theta + \frac{\theta^2}{2} + \frac{2\theta^3}{3} + \dots \infty$

(810) $\frac{2\theta}{1^2 - \theta^2} + \frac{2\theta}{3^2 - \theta^2} + \frac{2\theta}{5^2 - \theta^2} + \dots \infty$

Reference

$= \epsilon^{\sin\theta}$ L. 73

$= \epsilon^{\cos\theta}$ T. 126

$= \epsilon^{\theta}\sin\theta$ Y. 80

$= \operatorname{logh}\cos\theta$ Y. 107

$= \epsilon^{\sin^{-1}\theta}$ T. 126

$= \epsilon^{\theta}\cos\theta$ L. 79

$= \frac{1}{3}\left\{\epsilon^{\theta} + 2\epsilon^{-\theta/2}\cos\frac{\theta\sqrt{3}}{2}\right\}$ E. 190

$= \frac{1}{3}\epsilon^{\theta} - \frac{1}{3}\epsilon^{-\theta/2}\left\{\cos\frac{\theta\sqrt{3}}{2} - \sqrt{3}\sin\frac{\theta\sqrt{3}}{2}\right\}$ E. 190

$= \tan^{-1}\theta \quad \text{where } \theta^2 \geqslant 1$ T. 122

$= \sec^{-1}\theta = \frac{\pi}{2} - \operatorname{cosec}^{-1}\theta \quad \text{where } \theta > 1$ T. 122

$= \epsilon^{\tan\theta}$ T. 126

$= \epsilon^{\tan^{-1}\theta}$ T. 126

$= \epsilon^{\theta}\sec\theta$ Y. 107

$= \frac{\pi}{2}\tan\left\{\frac{\pi\theta}{2}\right\}$ A. 225

Series No.

(811) $\tan^{-1}\theta + \frac{h}{1+\theta^2} - \frac{\theta h^2}{(1+\theta^2)^2} - \frac{(1-3\theta^2)h^3}{3(1+\theta^2)^3} + \ldots \infty$

(812) $\sin^{-1}\theta + \frac{h}{\sqrt{1-\theta^2}} + \frac{\theta}{(1-\theta^2)^{3/2}}\frac{h^2}{2!} + \frac{1+2\theta^2}{(1-\theta^2)^{5/2}}\cdot\frac{h^3}{3!} + \ldots \infty$

(813) $\sec^{-1}\theta + \frac{h}{\theta\sqrt{\theta^2-1}} - \frac{2\theta^2-1}{\theta^2(\theta^2-1)^{3/2}}\cdot\frac{h^2}{2!} + \ldots \infty$

(814) $\frac{\theta}{1+\theta^2}\sum_0^\infty \frac{2^{2n}(n!)^2}{(2n+1)!}\left(\frac{\theta^2}{1+\theta^2}\right)^n$

(815) $\frac{1}{\theta} + \sum_{-\infty}^{+\infty} \frac{\theta}{n(\theta-n)} = \frac{1}{\theta} + \sum_1^\infty \frac{2\theta}{\theta^2-n^2}$

(816) $\frac{1}{n^2-1} - \frac{1}{3^3n^2-3} + \frac{1}{5^3n^2-5} - \frac{1}{7^3n^2-7} + \ldots \infty$

(817) $1 - \frac{n^2}{1^2} + \frac{n^2(n^2-1)}{1^2\cdot 2^2} - \frac{n^2(n^2-1^2)(n^2-2^2)}{1^2\cdot 2^2\cdot 3^2} + \ldots \infty$

(818) $\left(1+\frac{1}{1}\right) - \frac{\theta^2}{2!}\left(1-\frac{1}{3}\right) + \frac{\theta^4}{4!}\left(1+\frac{1}{5}\right) - \frac{\theta^6}{6!}\left(1-\frac{1}{7}\right) + \ldots \infty$

(819) $\frac{1}{4}\left[(1+3) - (3^2+3)\frac{\theta^2}{2} + \ldots + (-1)^n\{3^{2n}+3\}\frac{\theta^{2n}}{(2n)!} + \ldots \infty\right]$

(820) $\frac{\theta^2}{2} - \left(1+\frac{1}{3}\right)\frac{\theta^4}{4} + \left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{\theta^6}{6} - \ldots \infty$

(821) $\frac{1}{2}\cdot\frac{\theta^3}{3} - \left\{\frac{1}{2}+\frac{1}{4}\left(1+\frac{1}{3}\right)\right\}\frac{\theta^5}{5} + \left\{\frac{1}{2}+\frac{1}{4}\left(1+\frac{1}{3}\right)+\frac{1}{6}\left(1+\frac{1}{3}+\frac{1}{5}\right)\right\}\frac{\theta^7}{7} - \ldots \infty$

(822) $\sum_{-\infty}^{+\infty}\frac{1}{(\theta-n)^2} = \frac{1}{\theta^2} + 2\sum_1^\infty \frac{\theta^2+n^2}{(\theta^2-n^2)^2}$

Reference

$= \tan^{-1}(\theta + h)$ L. 71

$= \sin^{-1}(\theta + h)$ L. 71

$= \sec^{-1}(\theta + h)$ L. 71

$= \tan^{-1}\theta \quad \text{where } \theta^2 < \infty$ T. 122

$= \pi \cot \pi\theta \quad (n = 0 \text{ excluded})$ A. 217

$= \frac{\pi}{4}\left(\sec\frac{\pi}{2n} - 1\right)$ A. 225

$= \frac{\sin n\pi}{n\pi}$ C. 421

$= \cos\theta + \frac{\sinh\theta}{\theta}$ A. 389

$= \cos^3\theta$ Y. 79

$= \frac{1}{2}(\tan^{-1}\theta)^2 \quad \text{where } |x| < 1$ A. 191

$= \frac{(\tan^{-1}\theta)^3}{3!}$ Y. 89

$= \pi^2 \operatorname{cosec}^2 \pi\theta$ A. 218

Series No.

(823) $$\sum_{-\infty}^{+\infty} \frac{1}{(\theta - n)^3} = \frac{1}{\theta^3} + 2\sum_{1}^{\infty} \frac{\theta(\theta^2 + 3n^2)}{(\theta^2 - n^2)^3}$$

(824) $$\sum_{-\infty}^{+\infty} \frac{1}{(\theta - n)^4} = \frac{1}{\theta^4} + 2\sum_{1}^{\infty} \frac{\theta^4 + 6\theta^2 n^2 + n^4}{(\theta^2 - n^2)^4}$$

(825) $$\frac{1^4 - (1^2 - \theta^2)^2}{(1^2 - \theta^2)^2} + \frac{3^4 - (3^2 - \theta^2)^2}{(1^2 - \theta^2)^2(3^2 - \theta^2)^2} + \dots \infty$$

(826) $$\frac{1}{\theta^2} + \frac{1}{3} + \frac{1}{15}\theta^2 + \frac{2}{189}\theta^4 + \dots \infty$$

(827) $$\sum_{-\infty}^{+\infty} \frac{a - \theta}{(\theta - n)(a - n)}$$

(828) $$1 - \frac{n\theta^2}{3!} + \frac{n(5n - 2)}{3\cdot 5!}\theta^4 + \dots \infty$$

(829) $$\frac{1}{4}\left\{(3^3 - 3)\frac{\theta^3}{3!} + \dots (-1)^n \frac{3^{2n-1} - 3}{(2n - 1)!}\theta^{2n-1} - \dots \infty\right\}$$

(830) $$\frac{1}{\theta} + \frac{1}{\theta - 2} + \frac{1}{\theta + 2} + \frac{1}{\theta - 4} + \frac{1}{\theta + 4} + \dots \infty$$

(831) $$-\left(\frac{1}{\theta - 1} + \frac{1}{\theta + 1}\right) - \left(\frac{1}{\theta - 3} + \frac{1}{\theta + 3}\right) - \dots \infty$$

(832) $$\frac{1}{2}\frac{\theta^3}{3} + \left(\frac{1}{1^2} + \frac{1}{3^2}\right)\frac{1\cdot 3}{2\cdot 4}\frac{\theta^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\right)\frac{\theta^7}{7} + \dots \infty$$

(833) $$1 + a\theta + \frac{(a^2 + 1^2)\theta^2}{2!} + \frac{a(a^2 + 2^2)\theta^3}{3!} + \dots \infty$$

(834) $$\theta + \frac{2}{3}\theta^3 + \frac{2\cdot 4}{3\cdot 5}\theta^5 + \dots \infty$$

(835) $$\frac{mn}{m^2 + n^2}\left\{1 + \frac{2}{3}\cdot\frac{m^2}{m^2 + n^2} + \frac{2\cdot 4}{3\cdot 5}\left(\frac{m^2}{m^2 + n^2}\right)^2 + \dots \infty\right\}$$

Reference

$$= \pi^3 \cot \pi\theta \operatorname{cosec}^2 \pi\theta$$ A. 225

$$= \pi^4 \left\{\operatorname{cosec}^4 \pi\theta - \frac{2}{3} \operatorname{cosec}^2 \pi\theta\right\}$$ A. 225

$$= \tan^2 \frac{\pi\theta}{2}$$ C. 421

$$= \operatorname{cosec}^2 \theta$$ A. 222

$$= \pi (\cot \pi\theta - \cot \pi a)$$ A. 225

$$= \left(\frac{\sin \theta}{\theta}\right)^n$$ H. 498

$$= \sin^3 \theta$$ Y. 79

$$= \frac{\pi}{2} \cot \frac{\pi\theta}{2}$$ A. 225

$$= \frac{\pi}{2} \tan \frac{\pi\theta}{2}$$ A. 225

$$= \frac{1}{6} (\sin^{-1} \theta)^3 \quad \text{where } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$ A. 223

$$= \frac{\epsilon^{a \sin^{-1} \theta}}{\sqrt{1 - \theta^2}}$$ L. 79

$$= \frac{1}{\sqrt{1 - \theta^2}} \sin^{-1} \theta \quad \text{where } |\theta| < 1$$ A. 197

$$= \tan^{-1} \frac{m}{n}$$ A. 196

Series No.

(836) $\theta(1+\theta^2) - \frac{4}{1}\cdot\frac{\theta^3(1+\theta^2)^3}{3} + \frac{6\cdot 7}{1\cdot 2}\frac{\theta^5(1+\theta^2)^5}{5} - \frac{8\cdot 9\cdot 10}{1\cdot 2\cdot 3}\cdot\frac{\theta^7(1+\theta^2)^7}{7} + \ldots \infty$

(837) $1 - \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)(n-3)}{4!} - \ldots \infty$

(838) $(2^2-1)\frac{2}{2!}B_1\theta^2 + \frac{1}{2}(2^4-1)\frac{2^3}{4!}B_2\theta^4 + \frac{1}{3}(2^6-1)\frac{2^5}{6!}B_3\theta^6 + \ldots \infty$

(839) $1 + a\theta + \frac{a^2\theta^2}{2!} + \frac{a(a^2+1)}{3!}\theta^3 + \frac{a^2(a^2+2^2)}{4!}\theta^4 + \frac{a(a^2+1)(a^2+3^2)}{5!}\theta^5 + \ldots \infty$

(840) $\sum_{n=1}^{\infty} \frac{1}{(1-n^2m^2)^2}$

(841) $m\theta - \frac{m(m-1)(m-2)}{3!}\theta^3 + \frac{m(m-1)(m-2)(m-3)(m-4)}{5!}\theta^5 + \ldots \infty$

(842) $1 - \frac{m(m-1)}{2!}\theta^2 + \frac{m(m-1)(m-2)(m-3)}{4!}\theta^4 + \ldots \infty$

(843) $1 + a\theta + \frac{a^2-b^2}{2!}\theta^2 + \frac{a(a^2-3b^2)}{3!}\theta^3 + \frac{a^4-6a^2b^2+b^4}{4!}\theta^4 + \ldots + \frac{(a^2+b^2)^{n/2}}{n!}\theta^n \cos\left(n\tan^{-1}\frac{b}{a}\right) + \ldots \infty$

(844) $m\theta - \frac{m(m^2-1^2)}{3!}\theta^3 + \frac{m(m^2-1^2)(m^2-3^2)}{5!}\theta^5 - \ldots \infty$

Reference

$= \tan^{-1}\theta$ where $|\theta(1+\theta^2)|^2 < \frac{4}{27}$ A. 199

$= 2^{n/2}\cos\frac{n\pi}{4}$ where n is positive A. 311

$= -\ \text{logh}\cos\theta$ where $\theta^2 < \frac{\pi^2}{4}$ B. 245

$= \epsilon^{a\sin^{-1}\theta}$ L. 77

$= \frac{\pi^2}{4m^2}\text{cosec}^2\frac{\pi}{m} + \frac{\pi}{4m}\cot\frac{\pi}{m} - \frac{1}{2}$ A. 217, A. 218

$= \sin(m\tan^{-1}\theta)(1+\theta^2)^{m/2}$ L. 81

$= \cos(m\tan^{-1}\theta)(1+\theta^2)^{m/2}$ L. 81

$= \epsilon^{a\theta}\cos b\theta$ L. 73

$= \sin(m\sin^{-1}\theta)$ L. 76

Series No.

(845) $1 - \frac{m^2\theta^2}{2!} + \frac{m^2(m^2 - 2^2)}{4!}\theta^4 - \frac{m^2(m^2 - 2^2)(m^2 - 4^2)}{6!}\theta^6 + \dots \infty$

(846) $\sum_{-\infty}^{+\infty} \left\{\text{logh}\left(1 + \frac{\theta}{n}\right) - \frac{\theta}{n}\right\}$

(847) $\sum_{-\infty}^{+\infty} (-1)^n \left\{\text{logh}\left(1 + \frac{\theta}{n}\right) - \frac{\theta}{n}\right\}$

(848) $\theta^2 + \left(1 - \frac{1}{3} + \frac{1}{5}\right)\frac{\theta^6}{3} + \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}\right)\frac{\theta^{10}}{5} + \dots \infty$

(849) $S_2\frac{\theta^3}{3} - S_4\frac{\theta^5}{5} + S_6\frac{\theta^7}{7} - \dots \infty$

$$S_{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{2n}$$

(850) $\sum_{-\infty}^{+\infty} (-1)^n \left(\frac{1}{\theta - n} + \frac{1}{n}\right)$

(851) $\sum_{-\infty}^{+\infty} \left\{\frac{1}{n + x + (-1)^{n-1}y} - \frac{1}{n}\right\}$

In Series Nos. (852) to (854), see No. (1134) for values of $A_n(x)$ and $B_n(x)$.

(852) $2a\left(x - \frac{1}{2}\right) - \frac{(2a)^3}{2!}B_3(x) + \frac{(2a)^5}{4!}B_5(x) - \dots \infty$

(853) $-(2a)^2B_2(x) + \frac{(2a)^4}{3!}B_4(x) - \frac{(2a)^6}{5!}B_6(x) + \dots \infty$

(854) $1 - (2a)^2A_2(x) + \frac{(2a)^4}{3!}A_4(x) - \frac{(2a)^6}{5!}A_6(x) + \dots \infty$

(855) $\sum_{1}^{\infty} (-1)^{n-1}\frac{r!\theta^{2n-1}}{(2n - 1)!\,[r - (2n - 1)]!}$

See No. (841).

Reference

$= \cos(m \sin^{-1}\theta)$ L. 76

$= \operatorname{logh}\sin\pi\theta - \operatorname{logh}\pi\theta \quad \text{where } n \neq 0$ A. 314

$= \operatorname{logh}\tan\frac{\pi\theta}{2} - \operatorname{logh}\frac{\pi\theta}{2} \quad \text{where } n \neq 0$ A. 314

$= \frac{1}{2}\tan^{-1}\theta \operatorname{logh}\frac{1+\theta}{1-\theta}$ A. 191

$= \frac{1}{2}\tan^{-1}\theta \operatorname{logh}(1+\theta^2)$ A. 191

$= \pi \operatorname{cosec}(\pi\theta) - \frac{1}{\theta} \quad \text{omit } n = 0$

$= \frac{\pi\cos\pi x}{\sin\pi x - \sin\pi y} - \frac{1}{x-y} \quad \text{omit } n = 0$ A. 225

$= \frac{a\sin a(2x-1)}{\sin a}$ AE. 25

$= \frac{a\cos a(2x-1) - a\cos a}{\sin a}$ AE. 25

$= \frac{a\cos a(2x-1)}{\sin a}$ AE. 25

$= (1+\theta^2)^{r/2}\sin(r\tan^{-1}\theta)$ L. 81

Series No.

(856) $\sum_{1}^{\infty} (-1)^n \frac{r!\theta^{2n}}{(2n)!(r-2n)!}$

See No. (842).

(857) $\sum_{-\infty}^{+\infty} \frac{1}{n^4 + \theta^4}$

(858) $\sum_{-\infty}^{+\infty} \frac{1}{(n+x)^2 + y^2}$

(859) $\sum_{1}^{\infty} \frac{(-1)^n n}{(\epsilon^{n\pi} - \epsilon^{-n\pi})\{(n\pi)^4 + \frac{1}{4}\theta^4\}}$

In Series Nos. (860) to (862), see No. (1134) for values of $A_n x$.

(860) $2aA_1(x) - \frac{(2a)^3}{2!} A_3(x) + \frac{(2a)^5}{4!} A_5(x) + \ldots$

(861) $2a\left\{A_1(x) - 2A_1\left(\frac{x}{2}\right)\right\} - \frac{(2a)^3}{2!}\left\{A_3(x) - 2^3A_3\left(\frac{x}{2}\right)\right\}$
$+ \frac{(2a)^5}{4!}\left\{A_5(x) - 2^5A_5\left(\frac{x}{2}\right)\right\} - \ldots$

(862) $(2a)^2\left\{A_2(x) - 2^2A_2\left(\frac{x}{2}\right)\right\} - \frac{(2a)^4}{3!}\left\{A_4(x) - 2^4A_4\left(\frac{x}{2}\right)\right\}$
$+ \frac{(2a)^6}{5!}\left\{A_6(x) - 2^6A_6\left(\frac{x}{2}\right)\right\} - \ldots$

In series Nos. (863) to (873), see No. (330), etc. for values of p, q, r, and t.

(863) $p_1 + q_2a + p_3a^2 + q_4a^3 + \ldots$

(864) $r_1 + t_2a + r_3a^2 + t_4a^4 + \ldots$

Reference

$$= (1 + \theta^2)^{r/2} \cos (r \tan^{-1} \theta) - 1$$ L. 81

$$= \frac{\pi}{\theta^3\sqrt{2}} \frac{\sinh \pi\theta\sqrt{2} + \sin \pi\theta\sqrt{2}}{\cosh \pi\theta\sqrt{2} - \cos \pi\theta\sqrt{2}}$$ A. 313

$$= \frac{\pi}{y} \frac{\sinh 2\pi y}{\cosh 2\pi y - \cos 2\pi x}$$ A. 314

$$= \frac{1}{2\pi\theta^2 (\cosh \theta - \cos \theta)} - \frac{1}{2\pi\theta^4}$$ Q. 135

$$= \frac{a \sin (2x - 1)a}{\sin a}$$ AE. 93

$$= \frac{a \cos (2x - 1)a}{\cos a}$$ AE. 93

$$= \frac{a \sin (2x - 1)a}{\cos a}$$ AE. 93

$$= \frac{\pi}{4} \frac{1}{\sin \left(\frac{\pi}{4} - \frac{\pi a}{4}\right)}$$ AE. 80

$$= \frac{\pi}{6} \frac{1}{\sin \left(\frac{\pi}{6} - \frac{\pi a}{4}\right)}$$ AE. 80

Series No.

(865) $p_1 a + q_2 a^2 + p_3 a^3 + q_4 a^4 + \ldots$

(866) $r_1 a + t_2 a^2 + r_3 a^3 + t_4 a^4 + \ldots$

(867) $p_1 a + p_3 a^3 + p_5 a^5 + \ldots$

(868) $r_1 a + r_3 a^3 + r_5 a^5 + \ldots$

(869) $t_2 a^2 + t_4 a^4 + t_6 a^6 + \ldots$

(870) $q_2 a^2 + q_4 a^4 + q_6 a^6 + \ldots$

XVII. Hyperbolic Expansions

(871) $1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots \infty = \sum_0^\infty \frac{\theta^{2n}}{(2n)!}$

(872) $\theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots \infty = \sum_0^\infty \frac{\theta^{2n+1}}{(2n+1)!}$

Reference

$$= \frac{\pi a}{2} \frac{\sin\left(\frac{\pi}{4} + \frac{\pi a}{4}\right)}{\cos\frac{\pi a}{2}}$$ AE. 80

$$= \frac{2\pi a}{3} \frac{\sin\left(\frac{\pi}{6} + \frac{\pi a}{6}\right)}{2\cos\frac{\pi a}{3} - 1}$$ AE. 80

$$= \frac{\pi a}{2\sqrt{2}} \frac{\cos\frac{\pi a}{4}}{\cos\frac{\pi a}{2}}$$ AE. 81

$$= \frac{\pi a}{3} \frac{\cos\frac{\pi a}{6}}{2\cos\frac{\pi a}{3} - 1}$$ AE. 81

$$= \frac{\pi a}{\sqrt{3}} \frac{\sin\frac{\pi a}{6}}{2\cos\frac{\pi a}{3} - 1}$$ AE. 81

$$= \frac{\pi a}{2\sqrt{2}} \frac{\sin\frac{\pi a}{4}}{\cos\frac{\pi a}{2}}$$ AE. 81

$= \cosh\theta$ where $\theta^2 < \infty$ E. 84

$= \sinh\theta$ where $\theta^2 < \infty$ E. 84

Series No.

(873)† $\theta - \frac{\theta^3}{3} + \frac{2\theta^5}{15} - \frac{17\theta^7}{315} \ldots \infty$

$$= \sum_{1}^{\infty} (-1)^{n-1} \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_n \theta^{2n-1}$$

(874) $1 + \frac{m^2}{1^2} + \frac{m^2(m^2+1^2)}{1^2 \cdot 3^2} + \frac{m^2(m^2+1^2)(m^2+3^2)}{1^2 \cdot 3^2 \cdot 5^2} + \ldots \infty$

(875) $\theta + \frac{\theta^3}{3} + \frac{\theta^5}{5} + \ldots \infty$

(876) $\sum_{-\infty}^{+\infty} (-1)^{n-1} \frac{4n^2 - 1}{16n^4 + 4n^2 + 1}$

(878)‡ $1 + \sum_{1}^{\infty} (-1)^{n-1} \frac{2^{2n} B_n \theta^{2n}}{(2n)!} = 1 + \frac{\theta^2}{3} - \frac{\theta^4}{45} + \frac{2}{945}\theta^6 - \ldots \infty$

(879)† $1 + 2\sum_{1}^{\infty} (-1)^n \frac{(2^{2n-1} - 1) B_n \theta^{2n}}{(2n)!}$

$$= 1 - \frac{\theta^2}{6} + \frac{7\theta^4}{360} - \frac{31\theta^6}{15120} + \ldots \infty$$

(880) $\frac{1}{\theta} - 2\theta\left(\frac{1}{\pi^2+\theta^2} - \frac{1}{4\pi^2+\theta^2} + \frac{1}{9\pi^2+\theta^2} - \ldots \infty\right)$

(881) $4\pi\left(\frac{1}{\pi^2+4\theta^2} - \frac{3}{9\pi^2+4\theta^2} + \frac{5}{25\pi^2+4\theta^2} - \ldots \infty\right)$

(882) $\frac{1}{\theta} + 2\theta\left(\frac{1}{\pi^2+\theta^2} + \frac{1}{4\pi^2+\theta^2} + \frac{1}{9\pi^2+\theta^2} + \ldots\right)$

(883) $\sum_{1}^{\infty} \frac{(3^n - 3)\{1 - (-1)^n\}}{8n!} \theta^n$

(884) $\sum_{1}^{\infty} \frac{(3^n + 3)\{1 + (-1)^n\}}{8n!} \theta^n$

† For values of B_n, see No. (1129).

‡ No. (877) has been omitted because it duplicates a previous series.—Ed.

Reference

$= \tanh\theta \quad \text{where } \theta^2 < \frac{\pi}{4}$ H. 498

$= \cosh m\pi$ C. 421

$= \tanh^{-1}\theta \quad \text{where } |\theta| < 1$ H. 475

$= \pi\sqrt{2}\cosh\frac{\pi\sqrt{3}}{4}\operatorname{sech}\frac{\pi\sqrt{3}}{2}$ A. 314

$= \theta\coth\theta$ C. 343

$= \theta\operatorname{cosech}\theta \quad \text{where } \theta^2 < \pi^2$ C. 343

$= \operatorname{cosech}\theta$ Q. 136

$= \operatorname{sech}\theta$ Q. 136

$= \coth\theta$ Q. 136

$= \sinh^3\theta$ Y. 80

$= \cosh^3\theta$ Y. 80

Series No.

(885) $$\frac{b}{a^2+b^2} - \frac{b}{(2a)^2+b^2} + \frac{b}{(3a)^2+b^2} - \dots \infty$$
$$= \int_0^\infty \frac{\sin b\theta \, d\theta}{\epsilon^{a\theta}+1}$$

(886) $$\frac{b}{a^2+b^2} + \frac{b}{(2a)^2+b^2} + \frac{b}{(3a)^2+b^2} + \dots \infty$$
$$= \int_0^\infty \frac{\sin b\theta \, d\theta}{\epsilon^{a\theta}-1}$$

(887) $$1 - \frac{\theta^2}{3!} + \frac{14\theta^4}{6!} - \dots \infty$$

(888) $$\pi - 2\left(\theta - \frac{\theta^3}{3} + \frac{\theta^5}{5} - \dots\right)$$

(889) $$1 + \frac{1^2+1^2}{2!}\sin^2\theta + \frac{(1^2+1^2)(1^2+3^2)}{4!}\sin^4\theta + \dots \infty$$

(890) $$\frac{1^2}{1}\sin\theta + \frac{(1^2+2^2)}{3!}\sin^3\theta$$
$$+ \frac{(1^2+2^2)(1^2+4^2)}{5!}\sin^5\theta + \dots \infty$$

(891) $$1 + \frac{\theta^2}{2} - \frac{11\theta^4}{24} + \dots \infty$$

(892) $$\theta - \frac{\theta^3}{3} - \frac{\theta^5}{5} - \dots \infty$$

(893) $$1 - \frac{2^2\theta^4}{4!} + \frac{2^4\theta^8}{8!} - \dots \infty$$

(894) $$\frac{2^2\theta^2}{2!} - \frac{2^4\theta^6}{6!} + \frac{2^6\theta^{10}}{10!} - \dots \infty$$

(895) $$\theta + \frac{2\theta^3}{3!} - \frac{2^2\theta^5}{5!} - \frac{2^3\theta^7}{7!} + \frac{2^4\theta^9}{9!} + \dots \infty$$

(896) $$\theta - \frac{2\theta^3}{3!} - \frac{2^2\theta^5}{5!} + \frac{2^3\theta^7}{7!} + \dots \infty$$

Reference

$$= \frac{1}{2}\left[\frac{1}{b} - \frac{\pi}{a \sinh \dfrac{\pi b}{a}}\right]$$ A. 501

$$= \frac{\pi}{a}\left[\frac{1}{\epsilon^{2\pi b/a} - 1} - \frac{a}{2\pi b} + \frac{1}{2}\right]$$ A. 501

$$= \frac{\theta}{\sinh \theta}$$ H. 498

$$= \cos^{-1} (\tanh \operatorname{logh} \theta)$$ L. 80

$$= \frac{\cosh \theta}{\cos \theta}$$ L. 81

$$= \frac{\sinh \theta}{\cos \theta}$$ L. 81

$$= \cosh (\theta \cos \theta)$$ L. 80

$$= \sinh (\theta \cos \theta)$$ L. 80

$$= \cosh \theta \,.\, \cos \theta$$ H. 497

$$= 2 \sinh \theta \,.\, \sin \theta$$ T. 127

$$= \cosh \theta \,.\, \sin \theta$$ H. 497

$$= \sinh \theta \,.\, \cos \theta$$ H. 497

Series No.

(897) $1 - \frac{\theta^2}{2} + \frac{5}{24}\theta^4 - \frac{61}{720}\theta^6 + \ldots \infty = 1 + \sum_{1}^{\infty} (-1)^n \frac{E_n^*}{(2n)!}\theta^{2n}$

For E_n^* see No. (1131).

(898) $\sum_{0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2(2n+1)}\theta^{2n+1}$

$$= \operatorname{logh} 2\theta + \sum_{0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2 2n}\theta^{-2n}$$

(899) $\operatorname{logh} 2\theta - \sum_{0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2 2n}\theta^{-2n}$

$$= \operatorname{logh} 2\theta - \frac{1}{2}\cdot\frac{1}{2\theta^2} - \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{4\theta^4} - \ldots \infty$$

(900) $\sum_{0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2(2n+1)}\theta^{-2n-1}$

$$= \frac{1}{\theta} - \frac{1}{2}\cdot\frac{1}{3\theta^3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5\theta^5} - \ldots \infty$$

(901) $\frac{\theta^2}{2} + \frac{2}{3}\cdot\frac{\theta^4}{4} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{\theta^6}{6} - \ldots \infty$

For values of the coefficients in Nos. (902) through (938), see No. (1142).

(902) $2a\left(x - \frac{1}{2}\right) + \frac{(2a)^3}{2!}A_3(x) + \frac{(2a)^5}{4!}A_5(x) + \ldots \infty$

(903) $(2a)^2 B_2(x) + \frac{(2a)^4}{3!}B_4(x) + \frac{(2a)^6}{5!}B_6(x) + \ldots \infty$

(904) $1 + (2a)^2 A_2(x) + \frac{(2a)^4}{3!}A_4(x) + \frac{(2a)^6}{5!}A_6(x) + \ldots \infty$

(905) $(2a)^2 B_2\left(\frac{1}{4}\right) + \frac{(2a)^4}{3!}B_4\left(\frac{1}{4}\right) + \ldots \infty$

Reference

$= \operatorname{sech} \theta \quad \text{where } \theta^2 < \frac{\pi}{4}$ T. 127

where $\theta^2 < 1$

$= \sinh^{-1} \theta \quad \text{where } \theta^2 > 1$ T. 128

$= \cosh^{-1} \theta \quad \text{where } \theta^2 > 1$ T. 128

$= \sinh^{-1} \frac{1}{\theta} = \operatorname{cosech}^{-1} \theta \quad \text{where } \theta^2 > 1$ T. 128

$= \frac{(\sinh^{-1} \theta)^2}{2!}$ Y. 90

$= \frac{a \sinh a(2x - 1)}{\sinh a}$ AE. 19

$= \frac{a \cosh a(2x - 1) - a \cosh a}{\sinh a}$ AE. 6

$= \frac{a \cosh a(2x - 1)}{\sinh a}$ AE. 19

$= \frac{-2a \sinh \frac{1}{4}a \sinh \frac{3}{4}a}{\sinh a}$ AE. 33

Series No.

(906) $-\frac{1}{3}a + \frac{(2a)^3}{2!}B_3\left(\frac{1}{3}\right) + \frac{(2a)^5}{4!}B_5\left(\frac{1}{3}\right) + \dots \infty$

(907) $1 + (2a)^2A_2\left(\frac{1}{4}\right) + \frac{(2a)^4}{3!}A_4\left(\frac{1}{4}\right) + \dots \infty$

(908) $a - \frac{(4a)^3}{2!}B_3\left(\frac{1}{4}\right) - \frac{(4a)^5}{4!}B_5\left(\frac{1}{4}\right) - \dots \infty$

(909) $\frac{1}{2}a - \frac{(3a)^3}{2!}B_3\left(\frac{1}{3}\right) - \frac{(3a)^5}{4!}B_5\left(\frac{1}{3}\right) - \dots \infty$

(910) $2a - \frac{(6a)^3}{2!}B_3\left(\frac{1}{6}\right) - \frac{(6a)^5}{4!}B_5\left(\frac{1}{6}\right) - \dots \infty$

(911) $\frac{3a}{2} - \frac{(6a)^3}{2!}\left\{B_3\left(\frac{1}{6}\right) - \frac{1}{2^3}B_3\left(\frac{1}{3}\right)\right\}$
$- \frac{(6a)^5}{4!}\left\{B_5\left(\frac{1}{6}\right) - \frac{1}{2^5}B_5\left(\frac{1}{3}\right)\right\} - \dots \infty$

(912) $a - \frac{(6a)^3}{2!}B_3\left(\frac{1}{3}\right) - \frac{(6a)^5}{4!}B_5\left(\frac{1}{3}\right) - \dots \infty$

(913) $H_0 - \frac{H_1}{2!}(2a)^2 + \frac{H_2}{4!}(2a)^4 - \dots \infty$

(914) $2 - \frac{a^2}{2!}\left\{8^3B_3\left(\frac{1}{8}\right) - 4^3B_3\left(\frac{1}{4}\right)\right\}$
$- \frac{a^4}{4!}\left\{8^5B_5\left(\frac{1}{8}\right) - 4^5B_5\left(\frac{1}{4}\right)\right\} - \dots \infty$

(915) $1 + (2a)^2A_2\left(\frac{1}{4}\right) + \frac{(2a)^4}{3!}A_4\left(\frac{1}{4}\right) + \dots \infty$

(916) $\frac{1}{3} + (2-1)(3-1)B_1\frac{a^2}{2!} - (2^3-1)(3^3-1)B_2\frac{a^4}{4!}$
$+ (2^5-1)(3^5-1)B_3\frac{a^6}{6!} + \dots$

(917) $\frac{1}{3} - 2(3-1)B_1\frac{a^2}{2!} + 2^3(3^3-1)B_2\frac{a^4}{4!}$
$- 2^5(3^5-1)B_3\frac{a^6}{6!} + \dots$

Reference

$= -\dfrac{a \sinh \frac{1}{3}a}{\sinh a}$ AE. 35

$= \dfrac{a}{2 \sinh \frac{1}{2}a}$ AE. 33

$= \dfrac{a}{\cosh a}$ AE. 32

$= \dfrac{3a \sinh \frac{1}{2}a}{2 \ \sinh \frac{3}{2}a}$ AE. 51

$= \dfrac{3a \sinh 2a}{\sinh 3a}$ AE. 43

$= \dfrac{3a}{2} \dfrac{\cosh \frac{1}{2}a}{\cosh \frac{3}{2}a}$ AE. 51

$= \dfrac{3a \sinh a}{\sinh 3a}$ AE. 35

$= \dfrac{3 \cosh a}{2 \cosh 3a}$ AE. 49

$= \dfrac{2 \cosh a}{\cosh 2a}$ AE. 60

$= \dfrac{a}{2} \operatorname{cosech} \dfrac{a}{2}$ AE. 33

$= \dfrac{a \cosh 2a}{\sinh 3a}$ AE. 47

$= \dfrac{a \cosh a}{\sinh 3a}$ AE. 47

Series No.

(918) $1 + (2a)^2 A_2\left(\frac{1}{2}\right) + \frac{(2a)^4}{3!} A_4\left(\frac{1}{2}\right) + \frac{(2a)^6}{5!} A_6\left(\frac{1}{2}\right) + \dots$

(919) $a - \frac{(4a)^3}{2!} B_3\left(\frac{1}{4}\right) - \frac{(4a)^5}{4!} B_5\left(\frac{1}{4}\right) - \dots$

(920) $(2a)^2 B_2\left(\frac{1}{2}\right) + \frac{(2a)^4}{3!} B_4\left(\frac{1}{2}\right) + \frac{(2a)^6}{5!} B_6\left(\frac{1}{2}\right) + \dots$

(921) $1 + B_1 \frac{a^2}{2!} - B_2 \frac{a^4}{4!} + B_3 \frac{a^6}{6!} - \dots$

(922) $I_0 - \frac{I_1}{2!} a^2 + \frac{I_2}{4!} a^4 - \frac{I_3}{6!} a^6 + \dots$

(923) $(8a)^2 B_2\left(\frac{1}{4}\right) + \frac{(8a)^4}{3!} B_4\left(\frac{1}{4}\right) + \dots$

(924) $\frac{1}{2} a - \frac{(3a)^3}{2!} B_3\left(\frac{1}{3}\right) - \frac{(3a)^5}{4!} B_5\left(\frac{1}{3}\right) - \dots$

(925) $\frac{3}{2} a - \frac{a^3}{2!}\left\{6^3 B_6\left(\frac{1}{6}\right) - 3^3 B_3\left(\frac{1}{3}\right)\right\}$
$$- \frac{a^5}{4!}\left\{6^5 B_5\left(\frac{1}{6}\right) - 3^5 B_5\left(\frac{1}{3}\right)\right\} - \dots \infty$$

(926) $(12a)^2 B_2\left(\frac{1}{6}\right) + \frac{(12a)^4}{3!} B_4\left(\frac{1}{6}\right) + \frac{(12a)^6}{5!} B_6\left(\frac{1}{6}\right) + \dots \infty$

(927) $-(12a) B_2\left(\frac{1}{6}\right) - \frac{(12a)^3}{3!} B_4\left(\frac{1}{6}\right) - \frac{(12a)^5}{5!} B_6\left(\frac{1}{6}\right) - \dots \infty$

(928) $\frac{1}{2} a - \frac{(2a)^3}{2!} B_3\left(\frac{1}{4}\right) - \frac{(2a)^5}{4!} B_5\left(\frac{1}{4}\right) - \dots \infty$

(929) $\frac{1}{2} a - \frac{(3a)^3}{2!} B_3\left(\frac{1}{3}\right) - \frac{(3a)^5}{4!} B_5\left(\frac{1}{3}\right) - \dots \infty$

(930) $\frac{3}{4} a - \frac{(2a)^3}{2!} B_3\left(\frac{1}{8}\right) - \frac{(2a)^5}{4!} B_5\left(\frac{1}{8}\right) - \dots \infty$

(931) $(2a)^2 B_2\left(\frac{1}{3}\right) + \frac{(2a)^4}{3!} B_4\left(\frac{1}{3}\right) + \dots \infty$

	Reference
$= a \operatorname{cosech} a$	AE. 30
$= a \operatorname{sech} a$	AE. 32
$= - a \tanh \frac{1}{2} a$	AE. 30
$= \frac{1}{2} a \coth \frac{1}{2} a$	AE. 47
$= \frac{3(\sinh 2a - \sinh a)}{2 \sinh 3a}$	AE. 45
$= - \frac{8a \sinh a \sinh 3a}{\sinh 4a}$	AE. 33
$= \frac{3a}{2(1 + 2 \cosh a)}$	AE. 37
$= \frac{3a}{2(2 \cosh a - 1)}$	AE. 48
$= \frac{-12a \sinh a \sinh 5a}{\sinh 6a}$	AE. 46
$= \frac{\sinh 5a}{2(\cosh a + \cosh 3a + \cosh 5a)}$	AE. 46
$= \frac{a \sinh \frac{1}{2}a}{\sinh a}$	AE. 60
$= \frac{3a \sinh \frac{1}{2}a}{2 \sinh \frac{3}{2}a}$	AE. 48
$= \frac{a \sinh \frac{3}{4}a}{\sinh a}$	AE. 60
$= - \frac{2a \sinh \frac{1}{3}a \sinh \frac{2}{3}a}{\sinh a}$	AE. 40

Series No.

(932) $-\frac{2}{3}a + \frac{(2a)^3}{2!}B_3\left(\frac{1}{6}\right) + \frac{(2a)^5}{4!}B_5\left(\frac{1}{6}\right) + \ldots \infty$

(933) $-\frac{1}{3}a + \frac{(2a)^3}{2!}B_3\left(\frac{1}{3}\right) + \frac{(2a)^5}{4!}B_5\left(\frac{1}{3}\right) + \ldots \infty$

(934)† $t_2a^2 - t_4a^4 + t_6a^6 - \ldots \infty$

(935) $(2a)^2B_2\left(\frac{1}{4}\right) + \frac{(2a)^4}{3!}B_4\left(\frac{1}{4}\right) + \ldots \infty$

(936) $2a\left(x - \frac{1}{2}\right) + \frac{(2a)^3}{2!}A_3(x) + \frac{(2a)^5}{4!}A_5(x) + \ldots \infty$

(937) $1 + (2a)^2A_2(x) + \frac{(2a)^4}{3!}A_4(x) + \frac{(2a)^6}{5!}A_6(x) + \ldots \infty$

(938)† $p_1a - p_3a^3 + p_5a^5 - \ldots \infty$

(939) $\operatorname{logh}\frac{2}{\theta} - \sum_0^\infty \frac{(2n)!}{2^{2n}(n!)^2 2n}\theta^{2n}$

$$= \operatorname{logh}\frac{2}{\theta} - \frac{1}{2}\cdot\frac{\theta^2}{2} - \frac{1\cdot 3}{2\cdot 4}\cdot\frac{\theta^4}{4} - \ldots \infty$$

(940) $\operatorname{logh}\frac{2}{\theta} + \sum_0^\infty (-1)^n \frac{(2n)!}{2^{2n}(n!)^2 2n}\theta^{2n}$

$$= \operatorname{logh}\frac{2}{\theta} + \frac{1}{2}\cdot\frac{\theta^2}{2} - \frac{1\cdot 3}{2\cdot 4}\cdot\frac{\theta^4}{4} \ldots \infty$$

(941) $\sum_0^\infty \frac{\theta^{-2n-1}}{2n+1} = \frac{1}{\theta} + \frac{1}{3\theta^3} + \frac{1}{5\theta^5} + \ldots \infty$

(942) $\sum_0^\infty \epsilon^{-\theta(2n+1)}$

† For values of p and t, see No. (330).

Reference

$$= -\frac{a \sinh \frac{2}{3}a}{\sinh a}$$ AE. 43

$$= -\frac{a \sinh \frac{1}{3}a}{\sinh a}$$ AE. 35

$$= \frac{\pi a}{\sqrt{3}} \frac{\sinh \frac{\pi a}{6}}{2 \cosh \frac{\pi a}{3} - 1}$$ AE. 75

$$= -\frac{2a \sinh \frac{1}{4}a \sinh \frac{3}{4}a}{\sinh a}$$ AE. 33

$$= \frac{a \sinh a(2x - 1)}{\sinh a}$$ AE. 19

$$= \frac{a \cosh a(2x - 1)}{\sinh a}$$ AE. 19

$$= \frac{\pi a}{2\sqrt{2}} \cdot \frac{\cosh \frac{\pi a}{4}}{\cosh \frac{\pi a}{2}}$$ AE. 60

$$= \cosh^{-1} \frac{1}{\theta} = \operatorname{sech}^{-1} \theta \quad \text{where } \theta^2 < 1$$ T. 128

$$= \sinh^{-1} \frac{1}{\theta} = \operatorname{cosech}^{-1} \theta \quad \text{where } \theta^2 < 1$$ T. 128

$$= \tanh^{-1} \frac{1}{\theta} = \coth^{-1} \theta \quad \text{where } \theta^2 > 1$$ T. 128

$$= \frac{1}{2 \sinh \theta}$$ T. 129

Series No.

(943) $\sum_{0}^{\infty} (-1)^n \epsilon^{-\theta(2n+1)}$

(944) $\sum_{1}^{\infty} (-1)^n \epsilon^{-2n\theta}$

(945) $\sum_{0}^{\infty} \frac{1}{2n+1} \epsilon^{-\theta(2n+1)}$

(946) $\sum_{1}^{\infty} \left\{ \frac{1}{(2n\pi - a)^2 + \theta^2} + \frac{1}{(2n\pi + a)^2 + \theta^2} \right\}$

(947) $1 + \frac{2}{1 + 1^2} + \frac{2}{1 + 2^2} + \frac{2}{1 + 3^2} + \ldots \infty$

(948) $1 + \frac{2}{1 + 2^2} + \frac{2}{1 + 4^2} + \frac{2}{1 + 6^2} + \ldots \infty$

(949) $\frac{1}{2^2 + 1^2} + \frac{1}{2^2 + 3^2} + \frac{1}{2^2 + 5^2} + \ldots \infty$

(950) $\frac{1}{1 + 1^2} + \frac{1}{1 + 3^2} + \frac{1}{1 + 5^2} + \ldots \infty$

(951) $\frac{1}{2} + \theta^2 \sum_{1}^{\infty} \frac{1}{\theta^2 + n^2\pi^2}$

(952) $\sum_{-\infty}^{\infty} \frac{1}{(n + \theta)^2 + y^2}$

(953) $\frac{\theta^2}{6} - \frac{\theta^4}{180} + \ldots \infty$

$$= B_1^* \frac{2^2 \cdot \theta^2}{2 \cdot 2!} - B_3^* \frac{2^4 \cdot \theta^4}{4 \cdot 4!} + B_5^* \frac{2^6 \cdot \theta^6}{6 \cdot 6!} - \ldots \infty$$

(954) $1 + \frac{n\theta^2}{2!} + \frac{n(3n - 2)}{4!} \theta^4 + \ldots \infty$

(956)† $\theta - \frac{2}{3}\theta^3 + \frac{2 \cdot 4}{3 \cdot 5}\theta^5 - \ldots$

† No. (955) has been omitted because it duplicates a previous series.—Ed.

Reference

$= \dfrac{1}{2\cosh\theta}$ T. 129

$= \dfrac{1}{2}(\tanh\theta - 1)$ T. 129

$= -\dfrac{1}{2}\,\mathrm{logh}\tanh\dfrac{\theta}{2}$ T. 129

$= \dfrac{1}{2\theta}\cdot\dfrac{\sinh\theta}{\cosh\theta - \cos a} - \dfrac{1}{a^2 + \theta^2}$ A. 314

$= \pi\coth\pi$ $1 - i\psi(1+i) + i\psi(1-i)$ Y. 55

$= \dfrac{\pi}{2}\coth\dfrac{\pi}{2}$ Y. 55

$= \dfrac{\pi}{8}\tanh\pi$ Y. 55

$= \dfrac{\pi}{4}\tanh\dfrac{\pi}{2}$ Y. 55

$= \dfrac{\theta}{2}\coth\theta$ Y. 55

$= \dfrac{\pi}{y}\,\dfrac{\sinh 2\pi y}{\cosh 2\pi y - \cos 2\pi\theta}$ A. 314

$= \mathrm{logh}\,\dfrac{\sinh\theta}{\theta}$ Y. 109

$= \cosh^n\theta$ Y. 80

$= \dfrac{\sinh^{-1}\theta}{\sqrt{1 + \theta^2}}$ Y. 90

Series No.

XVIII. Taylor's and Maclaurin's Theorem

(957) $\phi(a) + x\phi'(a) + \frac{x^2}{2!}\phi''(a) \ldots \frac{x^n}{n!}\phi^n(a) + \ldots \infty$

(958) $\phi(0) + x\phi'(0) + \frac{x^2}{2!}\phi''(0) + \ldots \frac{x^n}{n!}\phi^n(0) + \ldots \infty$

XIX. Bessel Functions

(959) $\sum_{r=0}^{\infty} \frac{(-1)^r x^{n+2r}}{2^{n+2r} r!(n+r)!}$

(960) $1 - \frac{x^2}{2 \cdot 2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots \infty$

(961) $\frac{x}{2} - \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} - \frac{x^7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} + \ldots \infty$

(962) $\frac{x^n}{2^n \cdot n!}\left\{1 - \frac{x^2}{2^2 \cdot 1 \cdot (n+1)} + \frac{x^4}{2^4 \cdot 1 \cdot 2(n+1)(n+2)} - \ldots \infty\right\}$

(963) $\sum_{r=0}^{\infty} \frac{(-1)^r x^{n+2r}}{2^{n+2r} r! \Gamma(n+r+1)}$

(964) $\sqrt{\frac{2x}{\pi}}\left\{1 - \frac{x^2}{2 \cdot 3} + \frac{x^4}{2 \cdot 3 \cdot 4 \cdot 5} - \ldots \infty\right\}$

(965) $\left(\frac{2}{\pi x}\right)^{1/2} \cos x$

(966) $\frac{x^n \epsilon^{n\sqrt{1-x^2}}}{\sqrt{2\pi n}\,(1-x^2)^{1/4}\{1+\sqrt{1-x^2}\}^n}$

XX. Elliptic Functions

(967) $\frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \ldots \infty\right]$

(968) $\frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2 \frac{k^2}{1} - \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \frac{k^4}{3} - \ldots \infty\right]$

Reference

$= \phi(a + x)$ H. 481

$= \phi(x)$ H. 480

$= J_n(x)$ Q. 355

$= J_0(x)$ Q. 355

$= J_1(x)$ Q. 355

$= J_n(x)$ Q. 355

$= J_n(x)$ when n is any general value Q. 359

$= \left(\frac{2}{\pi x}\right)^{1/2} \sin x = J_{1/2}(x)$ Q. 364

$= J_{-1/2}(x)$ Q. 364

$\approx J_n(nx)$ when n is large and $0 < x < 1$ Q. 369

$= \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$ } where $0 < k < 1$ A. 190

$= \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta$ where $0 < k < 1$ A. 190

Series No.

XXI. Various Integrals

(969) $\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2}$

(970) $x \operatorname{logh} 2 + \frac{1}{2}\left\{\frac{\sin 2x}{1^2} + \frac{\sin 4x}{2^2} + \frac{\sin 6x}{3^2} + \ldots \infty\right\}$

(971) $x \operatorname{logh} 2 - \frac{1}{2}\left\{\frac{\sin 2x}{1^2} - \frac{\sin 4x}{2^2} + \frac{\sin 6x}{3^2} + \ldots \infty\right\}$

(972) $\frac{\sin 2x}{1^2} + \frac{\sin 6x}{3^2} + \frac{\sin 10x}{5^2} + \ldots \infty$

(973) $x - \frac{1}{3}x^3 + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \ldots \infty = \sum_0^\infty \frac{(-1)^k}{k!(2k+1)} x^{2k+1}$

(974) $x - \frac{x^5}{5\cdot 2!} + \frac{x^9}{9\cdot 4!} - \frac{x^{13}}{13\cdot 6!} + \ldots \infty$

$$= \sum_0^\infty \frac{(-1)^k}{(2k)!(4k+1)} x^{4k+1}$$

(975) $\frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \frac{1}{a+3b} + \ldots \infty$

(976) $\frac{1}{\epsilon}(1 - 2! + 3! - \ldots \infty) = \frac{0.4036526}{\epsilon}$

(977) $\operatorname{logh} x + \frac{ax}{1} + \frac{a^2x^2}{1\cdot 2^2} + \frac{a^3x^3}{1\cdot 2\cdot 3^2} + \ldots \infty$

(978) $\epsilon^{-ab}\left[\operatorname{logh}(b+x) + \frac{a(b+x)}{1\cdot 1} + \frac{a^2(b+x)^2}{2\cdot 2!} + \ldots \infty\right]$

(979) $\operatorname{logh} x + x \operatorname{logh} a + \frac{(x \operatorname{logh} a)^2}{2\cdot 2!} + \ldots \infty$

(980) $\frac{a^x x^n}{\operatorname{logh} a} - \frac{n a^x x^{n-1}}{(\operatorname{logh} a)^2} + \frac{n(n-1)a^x x^{n-2}}{(\operatorname{logh} a)^3} + \ldots$

$$\pm \frac{n(n-1)(n-2)\ldots 2\cdot 1\cdot a^x}{(\operatorname{logh} a)^{n+1}}$$

Reference

$$= \int_0^{\pi/2} \cos^n \theta \, d\theta = \int_0^{\pi/2} \sin^n \theta \, d\theta \quad \text{where } n \text{ is even}$$ J. 48

$$= -\int_0^x \text{logh} \sin x \, dx \quad \text{where } 0 < x < \pi$$ X. 142

$$= -\int_0^x \text{logh} \cos x \, dx \quad \text{where } -\frac{\pi}{2} < x < \frac{\pi}{2}$$ X. 142

$$= -\int_0^x \text{logh} \tan x \, dx \quad \text{where } 0 < x < \frac{\pi}{2}$$ X. 142

$$= \int_0^x \epsilon^{-x^2} \, dx \quad \text{where } x^2 < \infty$$

$$= \int_0^x \cos(x^2) \, dx \quad \text{where } x^2 < \infty$$

T. 133 See also A. 336 T. 134

$$= \int_0^1 \frac{x^{a-1} dx}{1 + x^b}$$ T. 134

$$= \int_1^\infty \frac{\epsilon^{-t}}{t} \, dt = \int_0^1 \frac{\epsilon^{-1/y}}{y} \, dy$$ A. 336

$$= \int \frac{\epsilon^{ax} \, dx}{x}$$

$$= \int \frac{\epsilon^{ax} \, dx}{b + x}$$

$$= \int \frac{a^x \, dx}{x}$$

$$= \int x^n a^x \, dx$$

Series No.

(981) $x + \frac{x^3}{3\cdot 3!} + \frac{7x^5}{3\cdot 5\cdot 5!} + \frac{31x^7}{3\cdot 7\cdot 7!} + \frac{127x^9}{3\cdot 5\cdot 9!} + \ldots \infty$

(982) $1 + \frac{x}{2^p} + \frac{x^2}{3^p} + \frac{x^3}{4^p} + \ldots \infty$

(983) $\frac{1}{1+1^2} + \frac{x}{1+2^2} + \frac{x^2}{1+3^2} + \ldots \infty$

(985)† $\operatorname{logh} \theta + \theta + \frac{\theta^2}{2\cdot 2!} + \frac{\theta^3}{3\cdot 3!} + \ldots \infty$

(986) $\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{2}{3}$

(987) $\theta - \frac{\theta^3}{3\cdot 3!} + \frac{\theta^5}{5\cdot 5!} - \frac{\theta^7}{7\cdot 7!} + \ldots \infty$

(988) $\operatorname{logh} \theta - \frac{\theta^2}{2\cdot 2!} + \frac{\theta^4}{4\cdot 4!} - \ldots \infty$

(989) $\frac{\pi}{2} \operatorname{logh} 2 = 2\sum_{1}^{\infty} (-1)^{n-1} \frac{\pi}{4n}$

(990) $2\sum_{0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$

(991) $\frac{\pi^2}{4} - 2\left[(1+a)\epsilon^{-a} + \frac{1}{3^2}(1+3a)\epsilon^{-3a} + \frac{1}{5^2}(1+5a)\epsilon^{-5a} + \ldots \infty\right]$

(992) $-C - \operatorname{logh}|x| + x - \frac{1}{2}\cdot\frac{x^2}{2!} + \frac{1}{3}\cdot\frac{x^3}{3!} - \frac{1}{4}\cdot\frac{x^4}{4!} + \ldots \infty$

C = Euler's constant; see No. (1132).

† No. (984) has been omitted because it duplicates a previous series.—Ed.

Reference

$$= \int \frac{x\,dx}{\sin x}$$ AA. 204

$$= \frac{1}{\Gamma(p)} \int_0^1 \frac{\left[\text{logh}\,\frac{1}{\zeta}\right]^{p-1}}{1 - x\zeta}\,d\zeta \quad \text{where } p > 0$$ A. 294 (1907)

$$= \int_0^1 \frac{\sin\left[\text{logh}\,\frac{1}{\zeta}\right]}{1 - x\zeta}\,d\zeta$$ A. 294 (1907)

$$= \int \frac{\epsilon^\theta}{\theta}\,d\theta$$ K. 305

$$= \int_0^{\pi/2} \cos^n\theta\,d\theta = \int_0^{\pi/2} \sin^n\theta\,d\theta \quad \text{where } n \text{ is odd}$$ 1Z. 226

$$= \int \frac{\sin\theta}{\theta}\,d\theta$$ K.305

$$= \int \frac{\cos\theta}{\theta}\,d\theta$$ K. 305

$$= \int_0^{\pi/2} \theta \cot\theta\,d\theta$$ 2Z. 187

$$= \int_0^{\pi/2} \frac{\theta\,d\theta}{\sin\theta}$$ See also No. (308). 2Z. 246

$$= \int_0^{\pi/2} \tan^{-1}(\sinh a \sin\theta)d\theta = \int_0^a \frac{\theta\,d\theta}{\sinh\theta} \quad \text{where } a \text{ is positive}$$ A. 517

$$= \int_x^\infty \frac{\epsilon^{-t}}{t}\,dt$$ A. 334

Series No.

(993) $\frac{1}{x} - \frac{1}{1!}\frac{1}{x+1} + \frac{1}{2!}\cdot\frac{1}{x+2} - \frac{1}{3!}\frac{1}{x+3} + \ldots \infty$

(994) $\sum_{0}^{\infty} \frac{1\cdot 3\ldots 2n-1}{2\cdot 4\ldots 2n}\cdot\frac{1}{(2n+1)^2}$

(995) $\sum_{0}^{\infty} (-1)^n \frac{1}{(2n+1)^2}$ See also No. (308).

(996) $\frac{x^2}{1\cdot 2} - \frac{x^3}{2\cdot 3} + \frac{x^4}{3\cdot 4} - \ldots \infty$

(997) $-\sum_{1}^{\infty} \frac{x^n}{n^2}$

(998) $\sum_{1}^{\infty} (-1)^{n-1} \frac{x^n}{n^2}$

(999) $2\sum_{1}^{\infty} \frac{x^{2n-1}}{(2n-1)^2}$

(1000) $\frac{1}{a} - \frac{1}{a+1} + \frac{1}{a+2} - \ldots \infty$

(1001) $\frac{1}{(n-1)!}\cdot\frac{\pi}{2^n}\Big[n^{n-1} - n(n-2)^{n-1} + \frac{n(n-1)}{2!}(n-4)^{n-1} + \ldots\Big]$

The number of terms in the brackets is $\frac{1}{2}n$ or $\frac{1}{2}(n+1)$.

(1002) $1 - \frac{x}{2^2} + \frac{x^2}{3^3} - \frac{x^3}{4^4} + \frac{x^4}{5^5} - \ldots \infty$

(1003) $\epsilon^{\theta} \sum_{r=0}^{r=n} (-1)^r \frac{n!}{(n-r)!}\, \theta^{n-r} 2^{-(r+1)/2} \sin\left\{\theta - \frac{(r+1)\pi}{4}\right\}$

(1004) $\frac{1}{\epsilon}\left\{1 + \frac{1}{3\cdot 1!} + \frac{1}{5\cdot 2!} + \frac{1}{7\cdot 3!} + \ldots \infty\right\}$

	Reference
$= \int_0^1 \epsilon^{-t} t^{x-1}\, dt$ where $x > 1$	Q. 260
$= \int_0^1 \frac{\sin^{-1} x}{x}\, dx$ where $\lvert x\rvert < 1$	AB. 165
$= \int_0^1 \frac{\tan^{-1} x}{x}\, dx$ where $\lvert x\rvert < 1$	AB. 165
$= \int_0^x \operatorname{logh}(1 + x)\, dx$ where $\lvert x\rvert < 1$	AB. 166
$= \int_0^x \operatorname{logh}(1 - x)\frac{dx}{x}$ where $0 < x < 1$	AB. 166
$= \int_0^x \operatorname{logh}(1 + \mathrm{x})\frac{dx}{x}$ where $0 < x < 1$	AB. 166
$= \int_0^x \operatorname{logh}\frac{1 + x}{1 - x}\cdot\frac{dx}{x}$ where $0 < x < 1$	AB. 166
$= \int_0^1 \frac{x^{a-1}}{1 + x}\, dx$ where $a > 0$	A. 189
$= \int_0^\infty \left(\frac{\sin\theta}{\theta}\right)^n d\theta$	
	A. 518
$= \int_0^1 \nu^{\nu x}\, d\nu$	1Z. 135
$= \int \theta^n \epsilon^\theta \sin\theta\, d\theta$	1Z. 135
$= \int_0^{\pi/2} \epsilon^{-\sin 2\theta} \cos\theta\, d\theta$	1Z. 638

Series No.

(1005) $1^2 + \left(\frac{1}{1!}\right)^2 + \left(\frac{1}{2!}\right)^2 + \left(\frac{1}{3!}\right)^2 + \ldots \infty$

(1006) $8\left[-\frac{a}{1\cdot 1\cdot 3} + \frac{a^3}{1\cdot 3\cdot 5} + \frac{a^5}{3\cdot 5\cdot 7} + \frac{a^7}{5\cdot 7\cdot 9} + \ldots \infty\right]$

(1007) $\frac{\pi}{2^n}\frac{\epsilon^{-ar}}{(n-1)!}\left[r^{n-1}a^{-n} + \frac{n(n-1)}{2}r^{n-2}a^{-(n+1)}\right.$
$\left. + \frac{(n+1)n(n-1)(n-2)}{2\cdot 4}r^{n-3}a^{-(n+2)} + \ldots n \text{ terms}\right]$

XXII. Beta and Gamma Functions. See also (1101).

(1008) $\int_0^1 x^{l-1}(1-x)^{m-1}\,dx$

$\int_0^\infty \epsilon^{-x}x^{n-1}\,dx$

$B(l, m)$

$\Gamma(n+1)$

$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right)$

$\Gamma(x)\Gamma\left(x+\frac{1}{n}\right)\Gamma\left(x+\frac{2}{n}\right)\ldots\Gamma\left(x+\frac{n-1}{n}\right)$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

(1009) $\sum_0^\infty \frac{2n!}{2^{2n}(n!)^2}\frac{1}{x+n}$

(1010) $\sum_0^\infty (-1)^n \frac{a(a-1)(a-2)\ldots(a-n)}{n!}\frac{1}{x+n}$

(1011) $\epsilon^{-x}x^{x-1/2}(2\pi)^{1/2}\left\{1 + \frac{1}{12x} + \frac{1}{288x^2}\right.$
$\left. - \frac{139}{51840x^3} - \frac{571}{2488320x^4} + 0\frac{1}{x^5}\right\}$

Reference

$$= \frac{2}{\pi}\int_0^{\pi} \epsilon^{2\cos\theta}\cos^2(\sin\theta)\,d\theta - 1$$ 2Z. 296

$$= \int_0^{\pi} \sin 2\theta \operatorname{logh}(1 - 2a\cos\theta + a^2)\,d\theta \quad \text{where } a^2 < 1$$ 2Z. 308

$$= \int_0^{\infty} \frac{\cos rx}{(a^2 + x^2)^n}\,dx$$ 2Z. 227

$= B(l, m)$ (Beta function)

$= \Gamma(n)$ (Gamma function)

$$= \frac{\Gamma(l)\Gamma(m)}{\Gamma(l + m)}$$

$= n\Gamma(n) = n!$ where n is a positive integer

$$= (2\pi)^{(n-1)/2}\, n^{-1/2}$$ 2Z. 62

$$= \Gamma(nx)(2\pi)^{(n-1)/2}n^{1/2-nx}$$ 2Z. 94

$$\Gamma(0) = \infty$$ 2Z. 59

$$= \frac{\Gamma(x)\Gamma(\frac{1}{2})}{\Gamma(x + \frac{1}{2})}$$ Q. 259

$$= \frac{\Gamma(x)\Gamma(a + 1)}{\Gamma(x + a)} \quad \text{where } a \text{ is positive}$$ Q. 260

$$= \Gamma(x)$$ Q. 253

Series No.

(1012) $\frac{1}{x} + \frac{1^2}{4} \cdot \frac{1}{x(x+1)} + \frac{1^2 \cdot 3^2}{4 \cdot 8} \frac{1}{x(x+1)(x+2)} + \dots \infty$

(1013) $\frac{1}{x} - \frac{a-1}{x+1} + \frac{(a-1)(a-2)}{2!(x+2)}$

$$- \frac{(a-1)(a-2)(a-3)}{3!(x+3)} + \dots \infty$$

(1014) $1 + \frac{a}{b}(x) + \frac{a(a+1)}{b(b+1)} x^2 + \frac{a(a+1)(a+2)}{b(b+1)(b+2)} x^3 + \dots \infty$

(1015) If in No. (1014), $b = 1$

$$1 + ax + \frac{a(a+1)}{2!} x^2 + \dots \infty$$

XXIII. Infinite Products

(1016) $\theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right) \dots \infty$

(1017) $\left(1 - \frac{4\theta^2}{\pi^2}\right)\left(1 - \frac{4\theta^2}{3^2\pi^2}\right)\left(1 - \frac{4\theta^2}{5^2\pi^2}\right) \dots \infty$

(1018) $2^{n-1}\left\{\cosh \phi - \cos \theta\right\}\left\{\cosh \phi - \cos\left(\theta + \frac{2\pi}{n}\right)\right\}$

$$\dots\left\{\cosh \phi - \cos\left(\theta + \frac{2n-2}{n}\pi\right)\right\}$$

$$= 2^{n-1} \prod_{r=0}^{r=n-1} \left\{\cosh \phi - \cos\left(\theta + \frac{2r\pi}{n}\right)\right\}$$

(1019) $\prod_{r=0}^{r=\frac{n-2}{2}} \left(x^2 - 2x \cos \frac{2r+1}{n} \pi + 1\right)$

(1020) $(x+1) \prod_{r=0}^{r=\frac{n-3}{2}} \left\{x^2 - 2x \cos \frac{2r+1}{n} \pi + 1\right\}$

(1021) $2^{(n-1)/2} \sin \frac{2\pi}{2n} \sin \frac{4\pi}{2n} \dots \sin \frac{n-2}{2n} \pi$

Reference

$$= \left[\frac{\Gamma(x)}{\Gamma(x+\frac{1}{2})}\right]^2 \quad \text{where } x \text{ is positive}$$ A. 524

$$= \frac{\Gamma(x)\Gamma(a)}{\Gamma(x+a)} \quad \text{where } x \text{ and } a \text{ are positive}$$ A. 524

$$= \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 \frac{t^{a-1}(1-t)^{b-a-1}}{1-xt}\,dt \quad \text{where } b > a > 0$$ A. 294 (1907)

$$= (1-x)^{-a}$$

$$= \sin\theta$$ A. 213

$$= \cos\theta$$ A. 214

$$= \cosh n\phi - \cos n\theta$$ E. 143

$$= x^n + 1 \quad \text{where } n \text{ is even}$$ E. 143

$$= x^n + 1 \quad \text{where } n \text{ is odd}$$ E. 143

$$= \sqrt{n} \quad \text{where } n \text{ is even}$$ E. 144

Series No.

(1022) $2^{(n-1)/2}\cos\frac{2\pi}{2n}\cos\frac{4\pi}{2n}\ldots\cos\frac{n-1}{2n}\pi$

(1023) $\cos\theta\cos\left(\theta+\frac{2\pi}{n}\right)\ldots\cos\left\{\theta+(n-1)\frac{2\pi}{n}\right\}$

(1024) $\sin\theta\sin\left(\theta+\frac{2\pi}{n}\right)\ldots\sin\left\{\theta+(n-1)\frac{2\pi}{n}\right\}$

(1025) $\tan\frac{\pi}{n}\tan\frac{2\pi}{n}\ldots\tan\frac{\frac{1}{2}(n-1)\pi}{n}$

(1026) $2^{n-1}\left(\cos\theta-\cos\frac{\pi}{2n}\right)\left(\cos\theta-\cos\frac{3\pi}{2n}\right)\ldots\left(\cos\theta-\cos\frac{2n-1}{2n}\pi\right)$

(1027) $2^{(n-1)/2}\sin\frac{\pi}{2n}\sin\frac{3\pi}{2n}\ldots\sin\frac{n-2}{2n}\pi$

$$=2^{(n-1)/2}\cos\frac{2\pi}{2n}\cos\frac{4\pi}{2n}\ldots\cos\frac{n-1}{2n}\pi$$

(1028) $(1-x)\left(1+\frac{1}{2}x\right)\left(1-\frac{1}{3}x\right)\left(1+\frac{1}{4}x\right)\ldots\infty$

(1029) $(1-x)\left(1+\frac{1}{3}x\right)\left(1-\frac{1}{5}x\right)\left(1+\frac{1}{7}x\right)\ldots\infty$

(1030) $\pi x\left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right)\left(1+\frac{x}{3}\right)\left(1-\frac{x}{1}\right)$

$$\times\left(1+\frac{x}{4}\right)\left(1+\frac{x}{5}\right)\left(1+\frac{x}{6}\right)\left(1-\frac{x}{2}\right)\ldots$$

(1031) $\prod_{-\infty}^{+\infty}\left\{\epsilon^{\theta/n\pi}\left(1-\frac{\theta}{n\pi}\right)\right\}$

Reference

$= 1$ where n is odd E. 144

$= \frac{1}{2^{n-1}} \cos n\theta$ where n is odd

$= \frac{1}{2^{n-1}} [(-1)^{n/2} - \cos n\theta]$ where n is even E. 73

$= (-1)^{(n-1)/2} \frac{1}{2^{n-1}} \sin n\theta$ where n is odd

$= (-1)^{n/2} \frac{1}{2^{n-1}} (1 - \cos n\theta)$ where n is even E. 73

$= \sqrt{n}$ where n is odd E. 145

$= \cos n\theta$ E. 145

$= 1$ where n is odd E. 145

$= \frac{\Gamma(\frac{1}{2})}{\Gamma(1 + \frac{1}{2}x)\Gamma(\frac{1}{2} - \frac{1}{2}x)}$ A. 115

$= \cos \frac{\pi x}{4} - \sin \frac{\pi x}{4}$ A. 224

$= e^{x \operatorname{logh} 3} \sin \pi x$ AG.40

$= \frac{\sin \theta}{\theta}$ omitting $n = 0$ A. 215

Series No.

(1032) $\prod_{n=1}^{n=\infty} \left\{\left(1 - \frac{\theta}{n\pi}\right)\epsilon^{\theta/n\pi}\left(1 + \frac{\theta}{n\pi}\right)\epsilon^{-\theta/n\pi}\right\}$

(1033) $\prod_{-\infty}^{+\infty} \left\{\epsilon^{2\theta/(2n+1)\pi}\left(1 - \frac{2\theta}{(2n+1)\pi}\right)\right\}$

(1034) $\prod_{2}^{\infty} \frac{n^3 - 1}{n^3 + 1}$

(1035) $\prod_{1}^{\infty} \frac{\left(1 + \frac{1}{r}\right)^x}{1 + \frac{x}{r}}$

(1036) $x!\left(x - \frac{1}{n}\right)!\left(x - \frac{2}{n}\right)! \ldots \left(x - \frac{n-1}{n}\right)!$

(1037) $\left\{\left(1 + \frac{k}{\theta}\right)^2\right\}\left\{1 + \left(\frac{k}{2\pi - \theta}\right)^2\right\}$

$\left\{1 + \left(\frac{k}{2\pi + \theta}\right)^2\right\}\left\{1 + \left(\frac{k}{4\pi - \theta}\right)^2\right\}\left\{1 + \left(\frac{k}{4\pi + \theta}\right)^2\right\} \ldots \infty$

(1038) $\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2^2}\cdot\cos\frac{\theta}{2^3} \ldots \infty$

(1039) $\theta\left(1 - \frac{\theta}{\pi}\right)\left(1 - \frac{\theta}{2\pi}\right)\left(1 + \frac{\theta}{\pi}\right)\left(1 - \frac{\theta}{3\pi}\right)$

$\times \left(1 - \frac{\theta}{4\pi}\right)\left(1 + \frac{\theta}{2\pi}\right) - \ldots \infty$

(1040) $\left(1 - \frac{\sin^2\theta}{\sin^2\beta}\right)\left(1 - \frac{\sin^2\theta}{\sin^2 3\beta}\right)\ldots\left(1 - \frac{\sin^2\theta}{\sin^2(n-1)\beta}\right)$

(1041) $2^{n-1} \prod_{0}^{n=1} \sin(\theta + ra)$

(1042) $\underset{n=\infty}{\mathrm{Lt}} \left[\tan\frac{\pi}{2n}\cdot\tan\frac{2\pi}{2n}\ldots\frac{\tan n\pi}{2n}\right]^{1/n}$

Reference

$= \dfrac{\sin\theta}{\theta}$ Q. 137

$= \cos\theta$ A. 216

$= \dfrac{2}{3}$

$= x!$ AD. 10

$= \sqrt{\dfrac{(2\pi)^{n-1}}{n}\,\dfrac{(n\cdot x)\,!}{n^{nx}}}$ AD. 18

$= \dfrac{\cosh k - \cos\theta}{1 - \cos\theta}$ Q. 137

$= \dfrac{\sin\theta}{\theta}$ A. 114

$= \epsilon^{-(\theta/\pi)\ \mathrm{logh}\ 2}\sin\theta$ Q. 35

$= \cos n\theta \quad (n \text{ even}) \quad \beta = \dfrac{\pi}{2n}$ A. 211

$= \sin n\theta \quad \text{where } 0 < \theta < a$ A. 211

$= 1$ 1Z. 355

Series No.

(1043) $\underset{n=\infty}{\mathrm{Lt}}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)^{1/2}\left(1+\frac{3}{n}\right)^{1/3}\ldots\left(1+\frac{n}{n}\right)^{1/n}\right]$

(1044) $\frac{x+c}{c}\prod_{-\infty}^{+\infty}\left(1+\frac{x}{n+c}\right)\epsilon^{-x/n}$

(1045) $2^{n-1}\prod_{r=0}^{r=n-1}\sin\left(\theta+\frac{r\pi}{n}\right)$

(1046) $2^{n-1}\sin\left(\theta+\frac{\pi}{2n}\right)\sin\left(\theta+\frac{3\pi}{n}\right)\ldots\sin\left(\theta+\frac{2n-1}{2n}\pi\right)$

(1047) $2^{n-1}\cos\theta\cos\left(\theta+\frac{\pi}{n}\right)\cos\left(\theta+\frac{2\pi}{n}\right)\ldots\left(\theta+\frac{n-1}{n}\pi\right)$

(1048) $2^{n-1}\cos\frac{\pi}{2n}\cos\frac{3\pi}{2n}\cos\frac{5\pi}{2n}\ldots\cos\frac{2n-1}{2n}\pi$

(1049) $2^{n-1}\sin\frac{\pi}{2n}\sin\frac{3\pi}{2n}\ldots\sin\frac{2n-1}{2n}\pi$

(1050) $\cos\frac{\pi}{n}\cos\frac{2\pi}{n}\ldots\cos\left(\frac{2n-1}{n}\right)\pi$

(1051) $\left(1-\frac{\theta}{a}\right)\left(1+\frac{\theta}{\pi-a}\right)\left(1-\frac{\theta}{\pi+a}\right)\left(1+\frac{\theta}{2\pi-a}\right)\left(1-\frac{\theta}{2\pi+a}\right)\ldots\infty$

(1052) $\prod_{0}^{\infty}\left\{1+\frac{\theta}{a+r\pi}\right\}$

(1053) $\prod_{0}^{\infty}\left\{1+\frac{2\theta}{2a+r\pi}\right\}$

(1054) $\prod_{0}^{\infty}\left\{1-\frac{2\theta}{2a+r\pi}\right\}$

Reference

$= \epsilon^{\pi^2/12}$ 1Z. 355

$= \dfrac{\sin \pi(x + c)}{\sin \pi c}$ A. 224

$= \sin n\theta$ E. 145

$= \cos n\theta$ E. 145

$= (-1)^{n/2} \sin n\theta$ where n is even

$= (-1)^{(n-1)/2} \cos n\theta$ where n is odd E. 145

$= \cos \dfrac{n\pi}{2}$ E. 146

$= 1$ E. 146

$= \dfrac{(-1)^n - 1}{2^{2n-1}}$ E. 146

$= \dfrac{\sin (a - \theta)}{\sin a}$ E. 159

$= \dfrac{\sin (a + \theta)}{\sin a}$ when r is a positive or negative integer or zero E. 159

$= \dfrac{\cos (a + \theta)}{\cos a}$ when r is an odd integer, positive or negative E. 159

$= \dfrac{\cos (a - \theta)}{\cos a}$ when r is an odd integer, positive or negative E. 159

Series No.

(1055) $\left\{1-\left(\dfrac{\theta}{\pi+a}\right)^2\right\}\left\{1-\left(\dfrac{\theta}{\pi-a}\right)^2\right\}\left\{1-\left(\dfrac{\theta}{3\pi+a}\right)^2\right\}$
$\times\left\{1-\left(\dfrac{\theta}{3\pi-a}\right)^2\right\}\ldots\infty$

(1056) $\left(1-\dfrac{\theta^2}{a^2}\right)\left\{1-\left(\dfrac{\theta}{2\pi+a}\right)^2\right\}\left\{1-\left(\dfrac{\theta}{2\pi-a}\right)^2\right\}\ldots\infty$

(1057) $\left(1-\dfrac{\theta}{a}\right)\left(1-\dfrac{\theta}{\pi-a}\right)\left(1+\dfrac{\theta}{\pi+a}\right)$
$\times\left(1+\dfrac{\theta}{2\pi-a}\right)\left(1-\dfrac{\theta}{2\pi+a}\right)\ldots\infty$

(1058) $\left(\dfrac{3^2}{3^2-1}\right)\left(\dfrac{5^2-1}{5^2}\right)\left(\dfrac{7^2}{7^2-1}\right)\left(\dfrac{9^2-1}{9^2}\right)\ldots\infty$

(1059) $\left(1-\dfrac{1}{3^2}\right)\left(1-\dfrac{1}{5^2}\right)\left(1-\dfrac{1}{7^2}\right)\ldots\infty$

(1060) $\displaystyle\prod_1^\infty \frac{n^3-x^3}{n^3+x^3}$

(1061) $\displaystyle\prod_1^\infty \frac{n(n+a+b)}{(n+a)(n+b)}$

(1062) $4\cos^2\dfrac{a}{2}\left\{1+\left(\dfrac{\theta}{\pi+a}\right)^2\right\}\left\{1+\left(\dfrac{\theta}{a-\pi}\right)^2\right\}\ldots\infty$

(1063) $\left(1+\dfrac{1}{1^2}\right)\left(1+\dfrac{1}{2^2}\right)\left(1+\dfrac{1}{3^2}\right)\ldots\infty$

(1064) $\displaystyle\prod_{m=1}^{m=\infty}\left\{1-\frac{x^2}{m^2-a^2}\right\}$

(1065) $\left(1+\dfrac{1}{1}\right)\left(1-\dfrac{1}{2}\right)\left(1+\dfrac{1}{3}\right)\left(1-\dfrac{1}{4}\right)\ldots\infty$

Reference

$$= \frac{\cos\theta + \cos a}{1 + \cos a}$$ E. 159

$$= \frac{\cos\theta - \cos a}{1 - \cos a}$$ E. 159

$$= \frac{\sin a - \sin\theta}{\sin a}$$ E. 160

$$= \frac{\{\Gamma(\frac{1}{4})\}^4}{16\pi^2}$$ Q. 259

$$= \frac{\pi}{4}$$ A. 106

$$= \frac{\Gamma(1+x)\Gamma(1+tx)\Gamma(1+t^2x)}{\Gamma(1-x)\Gamma(1-tx)\Gamma(1-t^2x)}$$

when $t = \frac{1}{2}(-1 + i\sqrt{3})$

or $t^3 = 1$ A. 313

$$= \frac{\Gamma(1+a)\Gamma(1+b)}{\Gamma(1+a+b)}$$ A. 115

$$= 2\cosh\theta + 2\cos a$$ E. 160

$$= \frac{1}{\pi}\sinh\pi$$ E. 160

$$= \frac{a}{\sqrt{a^2+x^2}}\,\frac{\sin\{\pi\sqrt{a^2+x^2}\}}{\sin\pi a}$$ E. 161

$$= 1$$ A. 108

Series No.

(1066) $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\infty$

(1067) $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\ldots\infty$

(1068) $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\infty$

(1069) $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{6}\right)\ldots\infty$

(1070) $\left(1+\frac{1}{1\cdot 3}\right)\left(1+\frac{1}{3\cdot 5}\right)$

(1071) $2(1+2^2)\left\{1+\left(\frac{2}{3}\right)^2\right\}\left\{1+\left(\frac{2}{5}\right)^2\right\}\ldots\infty$

(1072) $\left(1+\frac{1}{\sqrt{2}}\right)\left(1-\frac{1}{\sqrt{3}}\right)\left(1+\frac{1}{\sqrt{4}}\right)\ldots\infty$

$$=\left(1-\frac{1}{\sqrt{2}}\right)\left(1+\frac{1}{\sqrt{3}}\right)\left(1-\frac{1}{\sqrt{4}}\right)\ldots\infty$$

(1073)† $n\left(1-\frac{\sin^2\theta}{\sin^2\alpha}\right)\left(1-\frac{\sin^2\theta}{\sin^2 2\alpha}\right)\ldots\left\{1-\frac{\sin^2\theta}{\sin^2\frac{1}{2}(n-1)\alpha}\right\}$

(1074)† $n\left(1-\frac{\sin^2\theta}{\sin^2\alpha}\right)\left(1-\frac{\sin^2\theta}{\sin^2 2\alpha}\right)\ldots\left\{\frac{\sin^2\theta}{1\ \sin^2\frac{1}{2}(n-2)\alpha}\right\}$

(1075)‡ $\left(1-\frac{\sin^2\theta}{\sin^2\beta}\right)\left(1-\frac{\sin^2\theta}{\sin^2 3\beta}\right)\ldots\left\{1-\frac{\sin^2\theta}{\sin^2(n-2)\beta}\right\}$

(1075a)‡ $\left(1-\frac{\sin^2\theta}{\sin^2\beta}\right)\left(1-\frac{\sin^2\theta}{\sin^2 3\beta}\right)\ldots\left\{1-\frac{\sin^2\theta}{\sin^2(n-1)\beta}\right\}$

(1076) $2^{n-1}\prod_{0}^{n-1}\left\{\cos\theta-\cos\left(\beta+\frac{2\kappa\pi}{n}\right)\right\}$

† $\alpha=\frac{\pi}{n}$. ‡ $\beta=\frac{\pi}{2n}$.

Reference

$= \infty$ A. 106

$= 0$ C. 159

$= \frac{1}{2}$ A. 106

$= 0$ E. 155

$= \frac{\pi}{2}$ A. 213

$= \epsilon^{\pi} + \epsilon^{-\pi}$ E. 190

$= 0$ A. 111

$= \frac{\sin n\theta}{\sin \theta}$ where n is odd A. 210

$= \frac{\sin n\theta}{\sin \theta \cos \theta}$ where n is even A. 210

$= \frac{\cos n\theta}{\cos \theta}$ where n is odd A. 211

$= \cos n\theta$ where n is even A. 211

$= \cos n\theta - \cos n\beta$ T. 84

Series No.

(1077) $\prod_{0}^{n-1} \left\{a^2 - 2ab\cos\left(\theta + \frac{2\kappa\pi}{n}\right) + b^2\right\}$

(1078) $\theta \prod_{1}^{\infty} \left(1 + \frac{\theta^2}{n^2\pi^2}\right)$

(1079) $\prod_{0}^{\infty} \left\{1 + \frac{4\theta^2}{(2n+1)^2\pi^2}\right\}$

(1080) $\prod_{1}^{\infty} \cos\frac{\theta}{2^n}$

(1081) $\prod_{0}^{\infty} (1 + x^{2n})$

(1082) $\prod_{-\infty}^{\infty} \left[1 - \frac{x^2}{(n+c)^2}\right]$

(1083) $\prod_{-\infty}^{\infty} \left[1 - \frac{4x^2}{(n\pi + x)^2}\right]$

(1084) $\prod_{-\infty}^{\infty} \left[e^{\theta/n\pi}\left(1 - \frac{\theta}{n\pi}\right)\right]$

XXIV. Fourier's Series

(1085) $\frac{f'(2\pi) - f'(0)}{n^2} - \frac{f'''(2\pi) - f'''(0)}{n^4} + \dots \infty$

(1086) $\frac{1}{2}f(0) + f\left(\frac{2\pi}{n}\right) + f\left(\frac{4\pi}{n}\right) + \dots + \frac{1}{2}f(2\pi)$

(1087) $\frac{2\pi}{n}\left\{\frac{1}{2}f(0) + f\left(\frac{2\pi}{n}\right) + \dots + \frac{1}{2}f(2\pi)\right\}$

$$- 2\frac{f'(2\pi) - f'(0)}{n^2}\cdot\frac{\pi^2}{6} + 2\frac{f'''(2\pi) - f'''(0)}{n^4}\cdot\frac{\pi^4}{90} \dots \infty$$

Reference

$= a^{2n} - 2a^n b^n \cos n\theta + b^{2n}$ T. 84

$= \sinh \theta$ T. 130

$= \cosh \theta$ T. 130

$= \dfrac{\sin \theta}{\theta}$ T. 130

$= \dfrac{1}{1 - x}$ where $x^2 < 1$ T. 130

$= 1 - \dfrac{\sin^2 \pi x}{\sin^2 \pi c}$ A. 224

$= -\dfrac{\sin 3x}{\sin x}$ A. 224

$= \dfrac{\sin \theta}{\theta}$ $n = 0$ omitted A. 215

$= \int_0^{2\pi} f(t) \cos nt\, dt$ if all differential coefficients are finite and the series is convergent X. 140

$$= \frac{n}{2\pi} \int_0^{2\pi} f(t)\, dt + \frac{n}{\pi} \int_0^{2\pi} f(t) \cos nt\, dt + \frac{n}{\pi} \int_0^{2\pi} f(t) \cos 2nt\, dt + \dots \infty$$ X. 140

$= \int_0^{2\pi} f(t)\, dt$ where $\dfrac{\pi^2}{6} = 1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dots \infty$

and $\dfrac{\pi^4}{90} = 1 + \dfrac{1}{2^4} + \dfrac{1}{3^4} + \dots \infty$, etc. X. 140

Series No.

(1088) $$h\left\{\frac{1}{2}f(0) + f(h) + f(2h) + \dots \frac{1}{2}f(nh)\right\}$$
$$- \frac{h^2}{2\cdot 3!}\{f^1(nh) - f^1(0)\} + \frac{h^4}{6\cdot 5!}\{f^3(nh) - f^3(0)\}$$
$$- \frac{h^6}{6\cdot 7!}\{f^5(nh) - f^5(0)\} + \frac{3h^8}{10\cdot 9!}\{f^7(nh) - f^7(0)\}$$
$$- \frac{5h^{10}}{6\cdot 11!}\{f^9(nh) - f^9(0)\} + \dots \infty$$

Series to be convergent

(1089) $$h\left\{\frac{1}{2}f(x) + f(x+h) + f(x+2h) + \dots\right\} + \frac{h^2}{2\cdot 3!}f^1(x)$$
$$- \frac{h^4}{6\cdot 5!}f^3(x) + \frac{h^6}{6\cdot 7!}f^5(x) - \frac{3h^8}{10\cdot 9!}f^7(x) + \dots \infty$$

Series to be convergent

XXV. Hypergeometric Functions

(1090) $$1 + \frac{ab}{1\cdot c}x + \frac{a(a+1)b(b+1)}{2!c(c+1)}x^2$$
$$+ \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)}x^3 + \dots \infty$$

(1091) $$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

(1092) $F(A, 1; C, x)$

(1093) $$\frac{x^{m+1}}{m+1}\left\{1 + \frac{m+2}{m+3}x^2 + \dots \frac{(m+2)\dots(m+2n-2)}{(m+3)\dots(m+2n-1)}x^{2n-2}\right\}$$
$$+ (1-x^2)^{-1/2}\frac{(m+2)(m+4)\dots(m+2n)}{(m+1)(m+3)\dots(m+2n-1)}$$
$$\times \int_0^x t^{m+2n}(1-t^2) - \frac{1}{2}dt$$

Reference

$$= \int_0^{nh} f(t)\,dt$$ X. 140

$$= \int_x^{\infty} f(t)\,dt$$ X. 141

$$= F(a, b; c, x) \quad \text{where } |x| < 1$$ Q. 281

$$= F(a, b; c, 1)$$ Q. 282

$$= \frac{1}{1 - x} F\left(C - A, 1; C, \frac{x}{x - 1}\right)$$ Q. 286

$$= (1 - x^2)^{-1/2} \int_0^x t^m (1 - t^2)^{-1/2}\,dt$$ where x and $1 - x$ are not negative real numbers Q. 109

Series No.

XXVI. Relations between Products and Series

(1094) $(1 + xz)(1 + x^3z)\ldots(1 + x^{2m-1}z)$

(1095) $\text{logh}\,\{(1 - x)(1 - x^2)(1 - x^3)\ldots\ \infty\}$

(1096) $\prod_1^{\infty}\left(1 + \frac{x}{p_n}\right)$

(1097) $\prod_1^{m}(1 + x^n z)$

(1098) $\prod_1^{\infty}(1 + x^n z)$

(1099) In the following relations between series and products, all p are prime and may be related to the p_n, q_n, r_n, and t_n of series Nos. (315) to (318).

Series (315) $1 + \frac{1}{5^n} - \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n}\cdots = r_n$

Series (316) $1 - \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n}\cdots = t_n$

Series (317) $1 + \frac{1}{3^n} - \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{11^n}\cdots = p_n$

Series (318) $1 - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{11^n}\cdots = q_n$

Reference

$$= 1 + \sum_{n=1}^{m} \frac{(1 - x^{2m})(1 - x^{2m-2})\ldots(1 - x^{2m-2n+2})x^{n^2}z^n}{(1 - x^2)(1 - x^4)\ldots(1 - x^{2n})}$$ C. 344

$$= -\sum_{1}^{\infty} f(n)\frac{x^n}{n}$$ where $|x| < 1$ and $f(n)$ denotes the sum of all the divisors of the positive integer n; for example, $f(4) = 1 + 2 + 4$ C. 345

$$= 1 + \sum_{1}^{\infty} \frac{x(x + p_1)(x + p_2)\ldots(x + p_{n-1})}{p_1p_2p_3 \cdots p_n}$$ if the product converges to a definite limit C. 420

$$= 1 + \sum_{1}^{m} \frac{(1 - x^m)(1 - x^{m-1})\ldots(1 - x^{m-n+1})}{(1 - x)(1 - x^2)\ldots(1 - x^n)} x^{\{m(n+1)\}/2} z^n$$ C. 340

$$= 1 + \sum_{1}^{\infty} \frac{x^{\{n(n+1)\}/2}z^n}{(1 - x)(1 - x^2)\ldots(1 - x^n)}$$ where $|x| < 1$ C. 341

AE. 87

$$= \frac{1}{\left(1 - \frac{1}{5^n}\right)\left(1 + \frac{1}{7^n}\right)\left(1 + \frac{1}{11^n}\right)\left(1 - \frac{1}{13^n}\right)\cdots}$$

$$= \frac{1}{\left(1 + \frac{1}{5^n}\right)\left(1 + \frac{1}{7^n}\right)\left(1 - \frac{1}{11^n}\right)\left(1 - \frac{1}{13^n}\right)\cdots}$$

$$= \frac{1}{\left(1 - \frac{1}{3^n}\right)\left(1 + \frac{1}{5^n}\right)\left(1 + \frac{1}{7^n}\right)\left(1 - \frac{1}{11^n}\right)}$$ ... omitting non-primes

$$= \frac{1}{\left(1 + \frac{1}{3^n}\right)\left(1 + \frac{1}{5^n}\right)\left(1 - \frac{1}{7^n}\right)\left(1 + \frac{1}{11^n}\right)}$$ ... omitting non-primes

Series No.

(1100) $\prod_{1}^{\infty}(1 + v_n)$

XXVII. Special Functions

(1101) $\psi(x) = \dfrac{\Gamma'(x)}{\Gamma(x)} = \dfrac{d}{dx}\,\text{logh}\,\Gamma(x)$

C = Euler's constant; see No. (1132).

$$\psi(x) = -\frac{1}{x} - \frac{1}{x+1} - \frac{1}{x+2} - \cdots - \frac{1}{x+n} + \psi(x+n+1)$$

$$\sum_{0}^{\infty}\left(\frac{1}{x+n} - \frac{1}{n+1}\right)$$

$$\sum_{0}^{\infty}\left(\frac{1}{y+n} - \frac{1}{x+n}\right)$$

$$\sum_{1}^{\infty}\left(\frac{1}{n} - \frac{1}{x+n}\right)$$

$$\sum_{0}^{\infty}\frac{(-1)^n}{x+n}$$

$$\text{logh}\,n + \sum_{0}^{n-1}\frac{1}{n}\psi\left(1 + \frac{r}{n}\right)$$

$$\psi(x+1) = \frac{1}{x} + \psi(x)$$

$$\psi(1-x) = \psi(x) + \pi\cot\pi x$$

$$\psi(0) = -\infty$$

Reference

$$= 1 + \sum_{1}^{\infty} u_n \quad \text{where } \frac{v_n}{u_n} = \frac{1}{1 + u_1 + u_2 + \ldots u_{n-1}}$$ T. 131

and the series is convergent

$$= -C + \int_0^1 \frac{1 - t^{x-1}}{1 - t}\, dt$$ where x is a real and positive integer {2Z. 97–142, T. 132}

$$= -C + \frac{1}{1}\cdot\frac{x - 1}{x} + \ldots \frac{1}{n}\frac{x - 1}{x + n - 1} \ldots \infty$$ 2Z. 99

$$= -C - \psi(x)$$ T. 133

$$= \psi(x) - \psi(y)$$ A. 522

$$= \psi(x) + C + \frac{1}{x}$$ {2Z. 100, T. 132}

$$= \frac{1}{2}\left\{\psi\left(\frac{x + 1}{2}\right) - \psi\left(\frac{x}{2}\right)\right\} = \beta(x)$$ T. 133

$$= \int_0^1 \frac{t^x - 1}{t + 1}\, dt$$ A. 523

$$= \psi(n)$$ 2Z. 98

$$\psi\left(\frac{1}{2}\right) = -C - 2 \operatorname{logh} 2$$

$$\psi\left(\frac{3}{2}\right) = 2 - C - 2 \operatorname{logh} 2$$ A. 522

$$\psi\left(\frac{5}{2}\right) = \frac{8}{3} - C - 2 \operatorname{logh} 2$$

Series No.

$$\psi(1) = -C$$

$$\psi(2) = 1 - C$$

$$\psi(3) = \frac{3}{2} - C$$

$$\psi(4) = \left(1 + \frac{1}{2} + \frac{1}{3}\right) - C$$

$$\psi(n) = \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n}\right) - C$$

$$\beta(x + 1) + \beta(x) = \frac{1}{x}$$

$$\beta(x) + \beta(1 - x) = \frac{\pi}{\sin \pi x}$$

$$\beta(1) = \text{logh } 2$$

$$\beta(2) = 1 - \text{logh } 2$$

$$\beta(3) = -\frac{1}{2} + \text{logh } 2$$

$$\beta(4) = \frac{1}{3} + \frac{1}{2} - \text{logh } 2$$

$$\beta\left(\frac{1}{2}\right) = \frac{\pi}{2} \qquad \beta\left(\frac{1}{3}\right) = \text{logh } 2 + \frac{1}{3}\pi\sqrt{3}$$

$$\beta\left(\frac{3}{2}\right) = 2 - \frac{\pi}{2} \qquad \beta\left(\frac{2}{3}\right) = -\text{logh } 2 + \frac{1}{3}\pi\sqrt{3}$$

$$\beta\left(\frac{5}{2}\right) = \frac{2}{3} - 2 + \frac{\pi}{2}$$

If $\frac{d^2}{dx^2} \text{logh } \Gamma(x) = \psi'(x)$

Reference

2Z. 136

$$\psi\left(\frac{1}{3}\right) = -C - \frac{3}{2}\operatorname{logh} 3 - \frac{\pi}{2\sqrt{3}}$$

$$\psi\left(\frac{2}{3}\right) = -C - \frac{3}{2}\operatorname{logh} 3 + \frac{\pi}{2\sqrt{3}}$$

$$\psi\left(\frac{3}{4}\right) = \frac{\pi}{2} - C - 3\log 2$$

2Z. 136
A. 522

$$\psi\left(\frac{1}{4}\right) = -\frac{\pi}{2} - C - 3\log 2$$

T. 133

A. 523

$$= \sum_{0}^{\infty} \frac{1}{(x+n)^2}$$

2Z. 98

Series No.

$\psi'(0)$

$$\psi'\left(\frac{1}{2}\right) = 4\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right)$$

$$\psi'(1) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

$$\psi'\left(\frac{3}{2}\right) = 4\left(\frac{1}{3^2} + \frac{1}{5^2} + \ldots\right)$$

$$\psi'(2) = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

$$\psi'\left(\frac{5}{2}\right) = 4\left(\frac{1}{5^2} + \frac{1}{7^2} + \ldots\right)$$

$$\psi'(3) = \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots$$

$\psi'(\infty)$

(1102) If $\frac{a!}{b!(a-b)!'} = \binom{a}{b} = {}_aC_b$

$$\sum_{0}^{\frac{r}{2}-p} \binom{r}{2p+2n}\binom{p+n}{n}$$

$$\sum_{0}^{\frac{r-1}{2}-p} \binom{r}{2p+2n+1}\binom{p+n}{n}$$

$$1 + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$$

$$1 - \binom{n}{1} + \binom{n}{2} - \ldots (-1)^n \binom{n}{n}$$

$$1 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots \binom{n}{n}^2$$

Reference

$= \infty$ 2Z. 100

$= \frac{\pi^2}{2}$

$= \frac{\pi^2}{6}$

$= \frac{\pi^2}{2} - 4$

$= \frac{\pi^2}{6} - 1$

$= 4\left(\frac{\pi^2}{8} - \frac{1}{1^2} - \frac{1}{3^2}\right) = \frac{\pi^2}{2} - 4 \cdot \dot{4}$

$= \frac{\pi^2}{6} - \frac{5}{4}$

$= 0$

$= \frac{a(a-1)(a-2)\ldots(a-b+1)}{b!}$ T. 19

$= \frac{r}{2(r-p)}\binom{r-p}{p}2^{r-2p}$

$= \binom{r-p-1}{p}2^{r-2p-1}$

$= 2^n$ T. 19

$= 0$ T. 19

$= \binom{2n}{n}$ T. 19

Series No.

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \dots \binom{n}{k}$$

$$1 - \binom{n}{1} + \binom{n}{2} - \dots (-1)^k \binom{n}{k}$$

See also No. (189), etc.

Table of Binomial Coefficients $\binom{n}{1} = n^{C_1} = n$

$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\binom{n}{6}$	$\binom{n}{7}$	$\binom{n}{8}$	$\binom{n}{9}$	$\binom{n}{10}$	$\binom{n}{11}$	$\binom{n}{12}$
1											
2	1										
3	3	1									
4	6	4	1								
5	10	10	5	1							
6	15	20	15	6	1						
7	21	35	35	21	7	1					
8	28	56	70	56	28	8	1				
9	36	84	126	126	84	36	9	1			
10	45	120	210	252	210	120	45	10	1		
11	55	165	330	462	462	330	165	55	11	1	
12	66	220	495	792	924	792	495	220	66	12	1

XXVIII. Zeta Functions

(1103) $$\sum_{n=0}^{\infty} \frac{1}{(a+n)}^{s}$$

If s is a negative integer $= -m$

$\zeta(-2m)$

$\zeta(1 - 2m)$

$\zeta(0)$

Reference

$$= \binom{n+1}{k+1}$$ T. 19

$$= (-1)^k \binom{n-1}{k}$$ T. 19

T. 20

$$= \zeta(s, a)$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}\epsilon^{-ax}}{1 - \epsilon^{-x}}\, dx$$ Q. 266

$$= 0$$

$$= \frac{(-1)^m B_m}{2m} \quad \text{where } m = 1, 2, 3, \text{etc.}$$ Q. 268

$$= -\frac{1}{2}$$

Series No.

XXIX. Legendre Polynomials

(1104) If $P_0(x) + hP_1(x) + h^2P_2(x) + \ldots$

where $P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{1}{2}(3x^2 - 1)$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x),\ P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

and

$$P_n(x) = \frac{(2n)!}{2^n(n!)^2}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-4}\ldots\right]$$

then

$$\sum_{r=0}^{m}(-1)^r\frac{(2n-2r)!}{2^n r!(n-r)!(n-2r)!}x^{n-2r}$$

Also

$$P_n(1) = 1 \quad P_n(-1) = (-1)^n \quad P_{2n+1}(0) = 0$$

$$P_{2n}(0) = (-1)^n\frac{1\cdot 3\ldots(2n-1)}{2\cdot 4\ldots(2n)}$$

Finally, $P_n(x) = \omega$ is a solution of

$$P_n(x) = F\left(n+1, -n; 1; \frac{1}{2} - \frac{1}{2}x\right) = P_{-n-1}(x)$$

XXX. Special Products

(1105) If $q_0 = \prod_1^{\infty}(1 - q^{2n})$

$$q_1 = \prod_1^{\infty}(1 + q^{2n})$$

Reference

$= (1 - 2xh + h^2)^{-1/2}$ where $|2xh - h^2| < 1$ Q. 302

$= P_n(x)$ where $m = \frac{n}{2}$ or $\frac{n-1}{2}$ whichever is an integer

$$(1 - z^2)\frac{d^2\omega}{dx^2} - 2x\frac{d\omega}{dx} + n(n+1)\omega = 0$$ Q. 304

$$= \sum_{r=0}^{\infty} \frac{(n+1)(n+2)\ldots(n+r)(-n)(1-n)\ldots(r-1-n)}{(r!)^2} \times \left(\frac{1}{2} - \frac{1}{2}x\right)^r$$ Q. 312

Then $q_0 q_3 = \prod_1^{\infty}(1 - q^n)$ where $|q| < 1$ A. 116

$q_1 q_2 = \prod_1^{\infty}(1 + q^n)$ where $n = 1, 2, 3$, etc.

Series No.

$$q_2 = \prod_1^{\infty} (1 + q^{2n-1})$$

$$q_3 = \prod_1^{\infty} (1 - q^{2n-1})$$

The four products q_0, q_1, q_2, and q_3 are absolutely convergent.

For example, $$\frac{1}{(1 - q)(1 - q^3)(1 - q^5)\ldots\infty}$$

$$q_0 q_2^2 = 1 + 2q + 2q^4 + 2q^9 + \ldots\infty$$

$$q_0 q_1^2 = 1 + q^2 + q^6 + q^{12} + \ldots q^{n(n+1)} + \ldots\infty$$

$$\frac{q_0}{q_1} = 1 - 2q^2 + 2q^8 - 2q^{18} + \ldots + 2q^{2n^2} + \ldots\infty$$

$$q_0 q_3 = 1 - (q + q^2) + (q^5 + q^7) - (q^{12} + q^{15}) + \ldots\infty$$

$$q_2^8 = q_3^8 + 16 q q_1^8$$

$$q_0^3 = 1 - 3q^2 + 5q^6 - 7q^{12} + 9q^{20} \ldots$$

XXXI. General Forms

(1106) $$\sum_1^{\infty} \frac{\phi(n)x^n}{(n + a)(n + b)\ldots(n + k)}$$

If the degree of $\phi(n)$ is less than that of the denominator, resolve into partial fractions and use

$$\sum_1^{\infty} \frac{x^n}{a + n} = -x^{-a}\left\{\frac{x}{1} + \frac{x^2}{2} + \ldots + \frac{x^a}{a} + \text{logh}\,(1 - x)\right\}$$

See also No. (370).

Reference

$q_1q_2q_3 = 1$

$= (1 + q)(1 + q^2)(1 + q^3) \dots \infty$

$q_0q_3^2 = 1 - 2q + 2q^4 - 2q^9 + \dots \infty$

$q_0^3 = 1 - 3q^2 + 5q^6 - 7q^{12} + 9q^{20} - \dots \infty$

$\frac{q_0}{q_3} = 1 + q + q^3 + q^6 + q^{10} + \dots + q^{\{m(m+1)\}/2} + \dots \infty$

indices being alternately of the form $\frac{1}{2}n(3n \pm 1)$

This can always be summed if convergent. C. 246

$a, b, \dots k$ are positive or negative, unequal integers and $\phi(n)$ is an integral function of n.

If degree of $\phi(n)$ is greater than that of the denominator, it may be written

$$\psi(n)x^n + \frac{\chi(n)x^n}{(n + a)(n + b)\dots(n + k)}$$

where the degree of $\chi(n)$ is less than that of the denominator and

$$\sum \psi(n)x^n$$

may be found by partial fractions and (370)

Series No.

(1107) $\sum_{1}^{\infty} \frac{\phi(n)}{(n + a)(n + b)\ldots(n + k)}$

If in No. (1106) $x = 1$, the series is not convergent unless the degree of $\phi(n)$ is less than the degree of $(n + a)\ldots(n + k)$.

(1108) $\sum_{1}^{\infty} (-1)^{n-1} \frac{\phi(n)}{(n + a)(n + b)\ldots(n + k)}$

If absolutely convergent, Sum S_1

If semi-convergent, Sum S_2

In the above two, the series is absolutely convergent if the degree of $\phi(n)$ is less than $(n + a)\ldots(n + k)$ by two units, and semi-convergent if it is less than $(n + a)\ldots(n + k)$ by one unit.

(1109) $\sum_{0}^{\infty} \phi_r(n) \frac{x^n}{n!}$ can be summed

The identity can be established

$\phi_r(n)$

Then $\sum_{0}^{\infty} \frac{\phi_r(n)x^n}{n!}$

Reference

$$= - \sum_{a, b, c, \ldots k} \frac{\phi(-a)\left\{\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{a}\right\}}{(b - a)(c - a)\ldots(k - a)}$$ C. 248

where $\sum_{a, b, c \ldots k}$ means the summation with respect to

$a, b, c \ldots k$, and $a, b \ldots k$ are positive and unequal integers, and $\phi(n)$ is an integral function of n

$a, b \ldots k$ are all positive integers and $\phi(n)$ is an integral function of n. C. 253

$$= \sum_{a, b, c \ldots k} \frac{\phi(-a)\left\{\frac{1}{a} - \frac{1}{a-1} + \ldots (-1)^{a-1}\frac{1}{1}\right.}{(b - a)(c - a)\ldots(k - a)}$$

$$= \mathcal{S}_1 + \operatorname{logh} 2 \sum_{a, b, c \ldots k} \frac{(-1)^a \phi(-a)}{(b - a)(c - a)\ldots(k - a)}$$

if $\phi_r(n)$ is an integral function of n of the rth degree C. 234

(and see Part I, p. 107)

$$= A_0 + A_1 n + A_2 n(n - 1) + \ldots A_r n(n - 1)\ldots(n - r + 1)$$

$$= A_0 \sum_0^\infty \frac{x^n}{n!} + A_1 x \sum_1^\infty \frac{x^{n-1}}{(n-1)!} + A_2 x^2 \sum_2^\infty \frac{x^{n-2}}{(n-2)!} + \ldots$$

$$+ A_r x^r \sum_r^\infty \frac{x^{n-r}}{(x - r)!}$$

$$= (A_0 + A_1 x + A_2 x^2 + \ldots A_r x^r)e^x$$

Series No.

To determine the constants A_0, etc.,

divide $\phi_r(n)$ by n $\qquad\qquad \phi_r(n)$

divide $\phi_{r-1}(n)$ by $n - 1$ $\qquad\qquad \phi_{r-1}(n)$

$\phi_1(n)$

(1110) $\sum_0^\infty \phi_r(n)\, {}_mC_n\, x^n$

The identity can be established as in (1109)

$\phi_r(n)$

Then the general term in the series is

$$\phi_r(n)\, {}_mC_n\, x^n + A_0\, {}_mC_n\, x^n + A_1 n\, {}_mC_n\, x^n + \ldots + A_r n(n-1)\ldots(n-r+1)\, {}_mC_n\, x^n$$

Therefore

$$\sum_0^\infty \phi_r(n)\, {}_mC_n\, x^n = A_0 \sum_0^\infty {}_mC_n\, x^n + mA_1 x \sum_1^\infty {}_{m-1}C_{n-1}\, x^{n-1} + \ldots + m(m-1)\ldots(m-r+1)A_r x^r \sum_r^\infty {}_{m-r}C_{n-r}\, x^{n-r}$$

The constants A_0, etc., may be evaluated as in No. (1109).

(1111) $\dfrac{\sum_0^\infty \phi_r(n)\, {}_mC_n\, x^n}{(n+a)(n+b)\ldots(n+k)}$

This can in general be reduced to

$$\sum^\infty \frac{\psi(n)\, {}_{n+k}C_{m+k}\, x^{n+k}}{(m+1)(m+2)\ldots(m+k)x^k}$$

Reference

$$= \phi_{r-1}(n)n + A_0$$
$$= \phi_{r-2}(n)(n-1) + A_1$$
$$= A_r(n-r) + A_{r-1}$$

where $\phi_r(n)$ is any integral function of n of the rth degree C. 195

$$= A_0 + A_1 n + A_2 n(n-1) + \ldots + A_r n(n-1)\ldots(n-r+1)$$

$$\equiv A_0\,{}_mC_n\,x^n + mA_1 x\,{}_{m-1}C_{n-1}\,x^{n-1}$$
$$+ m(m-1)A_2x^2\,{}_{m-2}C_{n-2}\,x^{n-2} + \ldots$$
$$+ m(m-1)\ldots(m-r+1)A_rx^r\,{}_{m-r}C_{n-r}\,x^{n-r}$$

$$= A_0(1+x)^m + mA_1x(1+x)^{m-1} + \ldots$$
$$+ m(m-1)\ldots(m-r+1)A_rx^r(1+x)^{m-r}$$

$$= \left\{A_0 + \frac{mA_1x}{1+x} + \frac{m(m-1)A_2x^2}{(1+x)^2} + \ldots \right.$$
$$\left. + m(m-1)\ldots(m-r+1)\frac{A_rx^r}{(1+x)}r\right\}(1+x)^m$$

where $a, b \ldots k$ are unequal positive integers in ascending order of magnitude.

where $\psi(n)$ is an integral function of n viz. $\phi_r(n)$ multiplied by all the factors which are not absorbed by ${}_{m+k}C_{n+k}$.

C. 196

Series No.

(1112) $\sum a^x \phi(x)$

The series within the brackets stops at the nth difference of $\phi(x)$, supposing $\phi(x)$ to be of the nth degree.

$\phi(x)$ is rational and integral

(1113) Sum of a series whose nth term is r^n times an integral function of n of the $\mathcal{S}$th degree, such as

$$\sum^{n} \{a_0 n^{\mathcal{S}} + a_1 n^{\mathcal{S}-1} + \dots a_r\} r^n$$

Multiply by $(1 - r)^{\mathcal{S}+1}$. For example,

$$(1 - r)^3 \sum_{1}^{n} (1^2 r + 2^2 r^2 + \dots n^2 r^n)$$

whence by addition

$$\sum (1^2 r + 2^2 r^2 + \dots n^2 r^n)$$

(1114)† $\sum u_x$ (Approximate summation)

The constant K is to be determined in each case by substituting a known value of x.

(1115) If the sum $f(r)$ of a finite or infinite series

$f(r) = a_0 + a_1 r + a_2 r^2 + \dots$ is known, then

$$a_0 \cos x + a_1 r \cos (x + y) + a_2 r^2 \cos (x + 2y) \dots$$

$$a_0 \sin x + a_1 r \sin (x + y) + a_2 r^2 \sin (x + 2y) \dots$$

† For values of B_n^*, see No. (1129).

Reference

$$= C + \frac{a^x}{a-1}\left\{\phi(x) - \frac{a}{a-1}\Delta\phi(x) + \frac{a^2}{(a-1)^2}\Delta^2\phi(x) - \frac{a^3}{(a-1)^3}\Delta^3\phi(x)\ldots\right\}$$ W. 53

where $\Delta\phi(x) = \phi(x+h) - \phi(x)$

$$\begin{aligned} = 1^2r + 2^2r^2 + 3^2r^3 + \ldots\, n^2r^n \\ - 3r^2 - 3\cdot 2^2r^3 \ldots \\ + 3\cdot 1^2r^3 + \ldots \\ - 1^2r^4 \ldots - n^2r^{n+3} \end{aligned}$$

$$= \frac{r + r^2 - (n+1)^2r^{n+1} + (2n^2 + 2n - 1)r^{n+2} - n^2r^{n+3}}{(1-r)^3}$$

$$= K + \int u_x\,dx - \frac{1}{2}u_x + \frac{B_1^*}{2!}\frac{du_x}{dx} - \frac{B_3^*}{4!}\frac{d^3u_x}{dx^3} + \frac{B_5^*}{6!}\frac{d^5u_x}{dx^5} + \ldots\,\infty$$ 2Z. 117

$$= \frac{1}{2}\{e^{ix}f(re^{iy}) + e^{-ix}f(re^{-iy})\}$$

$$= -\frac{i}{2}\{e^{ix}f(re^{iy}) - e^{-ix}f(re^{-iy})\}$$ T. 81

Series No.

(1116) Euler's Summation Formula

$$\sum f(x) = f(1) + f(2) + \ldots + f(x)$$

When $f(x)$ is a rational algebraic fraction or a transcendental function, this cannot necessarily be used, and the right-hand side becomes an infinite series which may not converge. (See Bromwich, Chap. XII, for a number of applications of this summation.)

(1117) If $I_a\, I_a'\, I_a''$, etc., are the magnitudes of the differences of the discontinuities of the function and its various differential coefficients at point a, and similarly for points b, then

$$-\frac{1}{n}\sum I_a \sin na - \frac{1}{n^2}\sum I_a' \cos na + \frac{1}{n^3}\sum I_a'' \sin na + \frac{1}{n^4}\sum I_a''' \cos na - \ldots \infty$$

$$\frac{1}{n}\sum I_a \cos na - \frac{1}{n^2}\sum I_a' \sin na - \frac{1}{n^3}\sum I_a'' \cos na + \frac{1}{n^4}\sum I_a''' \sin na + \ldots \infty$$

XXXII. Double and Treble Series

(1118) $$\sum_0^\infty \sum_0^\infty \frac{(m+n)}{m!n!}\left(\frac{x}{2}\right)^{m+n}$$

(1119) $$\sum_0^\infty \sum_0^\infty \sum_0^\infty \frac{(m+n+p)!}{m!n!p!}\left(\frac{x}{3}\right)^{m+n+p}$$

Reference

$$= \int f(x)\,dx + \frac{1}{2}f(x) + \frac{1}{2!}B_1 f'(x) - \frac{1}{4!}B_2 f'''(x) + \dots$$ A. 304

where $f(x)$ is a polynomial and there is no term on the right-hand side (in its final form) which is not divisible by x.

$$= \pi a_n$$

(Fourier's nth harmonic amplitudes)

$$= \pi b_n$$ X. 57

where $a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt\,dt$ and $b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt\,dt$

$$\text{and } f(t) = \frac{a_0}{2} + a_1\cos t + a_2\cos 2t + a_3\cos 3t + \dots$$
$$+ b_1\sin t + b_2\sin 2t + b_3\sin 3t + \dots$$

$$= \frac{1}{1-x} \quad \text{where } -2 < x < 1$$ A. 194

$$= \frac{1}{1-x} \quad \text{where } -3 < x < 1$$ A. 194

Series No.

(1120) $$\sum_{r=2}^{\infty}\sum_{s=2}^{\infty}\frac{1}{(p+s)}r$$

(1121) $$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{1}{(4s-1)^{2r+1}}$$

(1122) $$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{1}{(4s-2)^{2r}}$$

(1123) $$\sum_{r=2}^{\infty}\sum_{s=1}^{\infty}\frac{1}{(2s)}r$$

(1124) $$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{1}{(4s-1)^{2r}}$$

(1125) $$\lim_{r=\infty}\left[\sum_{m=-r}^{r}\sum_{n=-r}^{r}\frac{1}{(x-m)(x-n)}\right]$$

(1126)† $$\sum_{n-1}^{n=\infty}\left[\sum_{m-1}^{m=\infty}\frac{\cos 2m\pi x\cdot\cos 2n\pi y}{m^2-n^2}\right]$$

(1127) $$\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{\infty}\frac{1}{(m^2+a^2)(n^2+b^2)}$$

XXXIII. Bernoulli's Functions

(1128) Bernoulli Functions.

Values of $\phi_n(x)$

$\phi_n(x)$

See also No. (1129).

† See No. (1128) for values of $\phi_2(y)$.

Reference

$$= \frac{1}{p+1}$$ A. 194

$$= \frac{\pi}{8} - \frac{1}{2} \operatorname{logh} 2$$ A. 194

$$= \frac{\pi}{8}$$ A. 194

$$= \operatorname{logh} 2$$ A. 194

$$= \frac{1}{4} \operatorname{logh} 2$$ A. 194

$$= -\pi^2 \quad \text{all values } m = n \text{ are excluded}$$ A. 225

$$= \pi^2 \left[\frac{3}{4}\{\phi_2(y) - \phi_2(x)\} + \frac{1}{4}\left(y - \frac{1}{2}\right)\right] \quad \text{omit } m = n$$ A. 391

$$= 0 \quad \text{if} \quad x = y \quad \text{where } 0 \leqslant y < x \leqslant \frac{1}{2}$$

$$= \frac{\pi^2}{ab} \coth \pi a \coth \pi b$$ Q. 136

$$= x^n - \frac{n}{2} x^{n-1} + \frac{n(n-1)}{2!} B_1 x^{n-2} - \frac{n(n-1)(n-2)(n-3)}{4!} B_2 x^{n-4}$$ A. 300

terminating either in x or x^2.

Series No.

Therefore

$$\phi_1(x) = x$$
$$\phi_2(x) = x^2 - x$$
$$\phi_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$$
$$\phi_4(x) = x^4 - 2x^3 + x^2$$
$$\phi_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x$$
$$\phi_6(x) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2$$
$$\phi_n(x+1) - \phi_n(x) = nx^{n-1}$$
$$\phi_2(x+1) = x^2 + x$$

etc.

This function is related to that in No. (1135) by $nB_n(x) = \phi_n(x)$.

(1129) Bernoulli's numbers can be calculated from the expression

$$\frac{2(2n)!}{(2\pi)^{2n}}\left\{\frac{1}{1^{2n}} + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \dots \infty\right\}$$

A similar confusion arises in the case of Euler's numbers, No. (1131). Some authorities, for the same value of n, quote the values of B_n, and others B_{2n-1}. In the tables following and in this book B_n values are used.

For example, in AC. 42, A. 297, C. 231 and AE. 3, etc.

$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42}, \text{ etc.}$$

whereas in Y. 503, etc.

$$B_1{}^* = \frac{1}{6}, \quad B_3{}^* = \frac{1}{30}, \quad B_5{}^* = \frac{1}{42}, \text{ etc.}$$

where $B_n = B_{2n-1}{}^*$.

Reference

$\phi_n(x)$ is the coefficient of $t_n/n!$ in the expansion of $t\,\dfrac{e^{xt}-1}{e^t-1}$ A. 300

C. 231
C. 363
A. 297
Y. 503

$$= \frac{2(2n)!}{[1-(\frac{1}{2})^{2n}](2\pi)^{2n}}\left\{\frac{1}{1^{2n}}+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\dots\infty\right\}$$

$$= B_n = B_{2n-1}{}^{*}$$

$$\frac{2(2n)!\,(-1)^{n+1}}{(2^{2n}-1)\,\pi^{2n}}\left[1+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\frac{1}{(2n+1)^{2n}}\right]$$

Extract from British Association Report, 1877 (Adams)

Table of Bernoulli's Numbers expressed in Vulgar Fractions

$B_0 = -1$

Numerator	Denominator	No.
1	6	1
1	30	2
1	42	3
1	30	4
5	66	5
691	2730	6
7	6	7
3617	510	8
43867	798	9
1 74611	330	10
8 54513	138	11
2363 64091	2730	12
85 53103	6	13
2 37494 61029	870	14
861 58412 76005	14322	15
770 93210 41217	510	16
257 76878 58367	6	17
26315 27155 30534 77373	1919190	18
2 92999 39138 41559	6	19
2 61082 71849 64491 22051	13530	20
15 20097 64391 80708 02691	1806	21
278 33269 57930 10242 35023	690	22
5964 51111 59391 21632 77961	282	23
560 94033 68997 81768 62491 27547	46410	24
49 50572 05241 07964 82124 77525	66	25

Table of Bernoulli's Numbers expressed in Integers and Repeating Decimals

No.	
1	$0.1\dot{6}$
2	$-0.0\dot{3}$
3	$0.0\dot{2}380\,9\dot{5}$
4	$-0.0\dot{3}$
5	$0.0\dot{7}\dot{5}$
6	$-0.2\dot{5}311\,3\dot{5}$
7	$1.1\dot{6}$
8	$-7.0\dot{9}215\,68627\,45098\,0\dot{3}$
9	$54.9\dot{7}117\,79448\,62155\,388\dot{4}$
10	$-529.1\dot{2}\dot{4}$
11	$6192.1\dot{2}318\,84057\,97101\,44927\,53\dot{6}$
12	$-86580.2\dot{5}311\,3\dot{5}$
13	$14\,25517.1\dot{6}$
14	$-272\,98231.0\dot{6}781\,60919\,54022\,98850\,57471\,264\dot{3}$
15	$6015\,80873.9\dot{0}064\,23683\,84303\,86817\,48359\,16771\,\dot{4}$
16	$-1\,51163\,15767.0\dot{9}215\,68627\,45098\,0\dot{3}$
17	$42\,96146\,43061.1\dot{6}$
18	$-1371\,16552\,05088.3\dot{3}277\,21590\,87948\,561\dot{6}$
19	$48833\,23189\,73593.1\dot{6}$
20	$-19\,29657\,93419\,40068.1\dot{4}863\,26681\,\dot{4}$
21	$841\,69304\,75736\,82615.0\dot{0}055\,37098\,56035\,43743\,07862\,67995\,57032\,11517\,16\dot{5}$
22	$40338\,07185\,40594\,55413.0\dot{7}681\,15942\,02898\,55072\,46\dot{3}$
23	$21\,15074\,86380\,81991\,60560.1\dot{4}539\,00709\,21985\,81560\,28368\,79432\,62411\,34751\,77304\,9\dot{6}$
24	$1208\,66265\,22296\,52593\,46027.3\dot{1}193\,70825\,25317\,81943\,54664\,94290\,02370\,17884\,07670\,760\dot{6}$
25	$75008\,66746\,07696\,43668\,55720.0\dot{7}\dot{5}$

In the "Report of the British Association for the Advancement of Science, 1877," page 10, etc., Bernoulli's numbers are calculated up to $n = 62$. Further, the method by which they are calculated is also described.

Series No.

Values of Constants used in Series Nos. (305) to (318)

(1130)

n	0	1	2	3	4	5
B_n	−1	1/6	1/30	1/42	1/30	5/66
α_n	0	1	1	3	17	155
β_n	1	1/3	7/15	31/21	127/15	2555/33
γ_n	0	1	2	13	164	3355
δ_n	2	1	13/5	121/7	1093/5	49205/11
ϵ_n	−1	1	91/5	3751/7	1 38811/5	251 43755/11
ζ_n	0	1	13	363	18581	15 25355
η_n	0	1	14	403	20828	17 14405
θ_n	0	1	10	273	13940	11 44055
E_n	{1	1	1	2	5	16
E_n^*	1*	1*	5*	61*	1385*	50521*
R_n	1	7	305	33367	68 15585	22374 23527
S_n	1	5	205	22265	45 44185	14916 32525
I_n	1/2	1/3	1	7	809/9	1847
H_n	3/2	3	33	903	46113	37 84503
J_n	2	10/3	34	910	4 15826/9	37 86350
T_n	0	1	23	1681	2 57543	676 37281
P_n	1	3	57	2763	2 50737	365 81523
Q_n	0	1	11	361	24611	28 73041

For the theory of Bernoulli polynomials, etc., including very many practically useful recurrence formulae, see Glaisher 1898a, c, and Lehmer 1935, 1936.

The numbers B_n, I_n, T_n, P_n, and Q_n are fundamental and are not expressible simply in terms of the others. It is to be noted that the odd Eulerian numbers are not calculable directly and are fundamental. See Edwards *Differential Calculus*, p. 502.

See No. (1131) for explanation of E_n and E_n^*.

Reference

AC. 42

Series No.

α_n

β_n

γ_n

δ_n

ϵ_n

η_n

ζ_n

θ_n

H_n

J_n

E_n^*

H_n

I_n

J_n

P_n

Q_n

See No. (330) for q_n.

† For a connection between $B_{2n+1}\left(\frac{1}{\kappa}\right)$, etc., see No. (1142).

Reference

$= 2(2^{2n} - 1)B_n$ AC. 41, 42, and 48

$= (2^{2n} - 2)B_n$

$= \frac{3}{4}(3^{2n} - 1)B_n$

$= (3^{2n} - 3)B_n$

$= \frac{1}{2}(2^{2n} - 2)(3^{2n} - 3)B_n$

$= \frac{3}{8}(2^{2n} - 2)(3^{2n} - 1)B_n$

$= \frac{1}{3}(2^{2n} - 1)(3^{2n} - 3)B_n$

$= \frac{1}{4}(2^{2n} - 1)(3^{2n} - 1)B_n$

$= (2^{2n+1} + 1)I_n$

$= 2(r^{2n} + 1)I_n$

$= H_n + I_n$

$= (-1)^{n+1}4^{2n+1}B_{2n+1}\left(\frac{1}{4}\right)^{\dagger}$ AE. 51

$= (-1)^{n+1}\left\{6^{2n+1}B_{2n+1}\left(\frac{1}{6}\right) - 3^{2n+1}B_{2n+1}\left(\frac{1}{3}\right)\right\}$ AE. 51

$= (-1)^{n+1}3^{2n+1}B_{2n+1}\left(\frac{1}{3}\right)$ AE. 51

$= (-1)^{n+1}6^{2n+1}B_{2n+1}\left(\frac{1}{6}\right)$ AE. 51

$= 2^{2n}E_n^* - (2n)_2 2^{2n-2}E_{n-1}^* \ldots (-1)^{n-1}(2n)_2 2^2 E_1^* + (-1)^n E_0^*$

$= \frac{1}{2}\left[(-1)^{n+1}8^{2n+1}B_{2n+1}\left(\frac{1}{8}\right) - E_n^*\right]$ AE. 62

$= \frac{q_n}{\sqrt{2}}(2n - 1)!\left(\frac{4}{\pi}\right)^{2n}$ AE. 64

Series No.

R_n

S_n

T_n

(1131) Euler's Numbers are calculated from the equation

$$2(2n)!\left(\frac{2}{\pi}\right)^{2n+1}\left\{\frac{1}{1^{2n+1}}-\frac{1}{3^{2n+1}}+\frac{1}{5^{2n+1}}-\ldots\infty\right\}$$

$$=2(2n)!\left(\frac{2}{\pi}\right)^{2n+1}\left(1+\frac{1}{3^{2n+1}}\right)\left(1-\frac{1}{5^{2n+1}}\right)\left(1+\frac{1}{7^{2n+1}}\right)\ldots$$

In the list below of Euler's numbers odd values of E_n have been included for completeness. These have sometimes been called "Prepared Bernoulli Numbers," and have been omitted from many lists, and only the even numbers included. This has led to confusion because in some cases the even numbers were then called "Euler's Numbers E_n."

In 2Z. 243 and Y. 501 Eulers' numbers are as shown in the long list below; but in AE., AC. 42, T. 141, and C. 342 and 365, the numbers are shown as $E_n{}^*$ where $E_n{}^* = E_{2n}$

$n =$	0	1	2	3	4	5	
$E_n{}^* =$	1	1	5	61	1385	50521	etc.

and throughout this collection E_n and $E_n{}^*$ have been used to distinguish between them.

$E_1 = 1 \quad E_2 = 1 \quad E_3 = 2 \quad E_4 = 5 \quad E_5 = 16$

$E_6 = 61 \quad E_7 = 272 \quad E_8 = 1385 \quad E_9 = 7936 \quad E_{10} = 50521$

(1132) Euler's Constant

$$C = 0.57721\ 56649\ 01532\ 86060\ldots$$

Reference

$$= \frac{1}{4}(3^{2n+1} + 1)E_n^*$$ AC. 50

$$= \frac{1}{2}(3^{2n} + 1)E_n^*$$ AC. 50

$$3S_n = 2R_n + E_n^*$$

$$= \frac{t_{2n}}{2\sqrt{3}}\cdot(2n-1)!\left(\frac{6}{\pi}\right)^{2n}$$ AE. 75

C. 342
C. 365
Y. 502

$$= E_n^* = E_{2n}$$

$E_{12} = 270\,2765$ $E_{14} = 1993\,60981$ $E_{16} = 1\,93915\,12145$
$E_{18} = 240\,48796\,75441$ $E_{20} = 37037\,11882\,37525$

2Z. 245

2Z. 87

Series No.

(1133) Sum of Power Series

$$\mathcal{S}_n = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \dots \infty$$

Values of $\mathcal{S}_n$ to sixteen places of decimals are given in the table.

n	$\mathcal{S}_n$ to sixteen places of decimals
1	0.57721 56649 01532 8606065 log ∞ (Euler's Const. + ∞)
2	1.64493 40668 48226 4
3	1.20205 69031 59594 3
4	1.08232 32337 11138 2
5	1.03692 77551 43370 0
6	1.01734 30619 84449 1
7	1.00834 92773 81922 7
8	1.00407 73561 97944 3
9	1.00200 83928 26082 2
10	1.00099 45751 27818 0
11	1.00049 41886 04119 4
12	1.00024 60865 53308 0
13	1.00012 27133 47578 5
14	1.00006 12481 35058 7
15	1.00003 05882 36307 0
16	1.00001 52822 59408 6
17	1.00000 76371 97637 9
18	1.00000 38172 93265 0
19	1.00000 19082 12716 6
20	1.00000 09539 62033 9
21	1.00000 04769 32986 8
22	1.00000 02384 50502 7
23	1.00000 01192 19926 0
24	1.00000 00596 08189 1
25	1.00000 00298 03503 5
26	1.00000 00149 01554 8
27	1.00000 00074 50711 8

The sixteenth decimal place is not always the sixteenth occurring, but the nearest in consideration of terms to follow, e.g. C has for its 16th, 17th, etc.... figures 8606....

$\text{Log}_e\, 10 = 2.30258\ 50929\ 94045\ 6840\dots$

Euler's Const. $= C = 0.57721\ 56649\ 01532\ 8606\dots$

$\mu = 0.43429\ 44819\dots$

Reference

2Z. 144

Series No.

n	S_n to sixteen places of decimals
28	1.00000 00037 25334 0
29	1.00000 00018 62659 7
30	1.00000 00009 31327 4
31	1.00000 00004 65662 9
32	1.00000 00002 32831 2
33	1.00000 00001 16415 5
34	1.00000 00000 58207 7
35	1.00000 00000 29103 8

Relations between Bernoulli's Numbers

(1134) $1^{n-1} + 2^{n-1} + 3^{n-1} + \dots (x-1)^{n-1}$

(1135) $$\frac{x^n}{n} - \frac{1}{2}x^{n-1} + \frac{n-1}{2!}B_1x^{n-2} - \frac{(n-1)(n-2)(n-3)}{4!}B_2x^{n-4} + \dots$$

The last term is

The last term is

(1136) $B_{2n+1}(1-x)$

$B_{2n}(1-x)$

$B_n(0)$

$B_n(1)$

(1137) $$B_n(x) - (n-1)xB_{n-1}(x) + (n-1)_2x^2B_{n-2}(x) + \dots (-1)^{n-2}x^{n-2}B_2(x) + (-1)^{n-1}x^n$$

$$(n)_r = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$$

(1138) $1^{2n} + 2^{2n} + 3^{2n} + \dots (2x)^{2n}$

$1^{2n} + 3^{2n} + \dots (2x-1)^{2n}$

$2^{2n} + 4^{2n} + \dots + (2x)^{2n}$

Reference

$= B_n(x)$ where x is a positive integer AE. 8

A. 304

$= B_n(x)$

$(-1)^{n/2}\frac{n-1}{2}B_{n/2-1}x^2$ where n is even

$(-1)^{(n-1)/2}B_{(n-1)/2}x$ where n is odd AE. 7

$= -B_{2n+1}(x)$ AE. 4

$= B_{2n}(x)$

$= 0$

$= 0$

$= -B_n(-x)$ AE. 11

$= {}_nC_r = \binom{n}{r}$

$= B_{2n+1}(2x+1)$

$= B_{2n+1}(2x+1) - 2^{2n}B_{2n+1}(x+1)$ AE. 15

$= 2^{2n}B_{2n+1}(x+1)$

Series No.

(1139) $nA_n(x)$

$A_{2n}(x)$

$A_n(1 - x)$

$A_{2n+1}(0)$

$A_{2n}(0)$

(1140) k is a positive integer

$$B_n(x) + B_n\left(x + \frac{1}{k}\right) + \ldots B_n\left(x + \frac{k-1}{k}\right)$$

(1141) $B_n\left(\frac{1}{k}\right) + B_n\left(\frac{2}{k}\right) + \ldots B_n\left(\frac{k-1}{k}\right)$

(1142) $B_{2n}\left(\frac{1}{6}\right)$

$B_{2n}\left(\frac{1}{4}\right)$

$B_{2n}\left(\frac{1}{3}\right)$

$B_{2n}\left(\frac{1}{2}\right)$

$B_{2n+1}\left(\frac{1}{12}\right)$

$B_{2n+1}\left(\frac{1}{8}\right) - B_{2n+1}\left(\frac{3}{8}\right)$

Reference

$= x^n - \frac{1}{2} nx^{n-1} + (n)_2 B_1 x^{n-2} - (n)_4 B_2 x^{n-4}$, etc., the series being continued so long as the exponents are not negative AE. 19

$= B_{2n}(x) + (-1)^{n-1} \frac{B_n}{2n}$ AE. 18 and 20

$= (-1)^n A_n(x)$

$= A_{2n+1}(1) = 0$

$= A_{2n}(1) = (-1)^{n-2} \frac{B_n}{2n}$

$= \frac{1}{k^{n-1}} B_n(kx)$ where n is odd AE. 9

$= \frac{1}{k^{n-1}} B_n(kx) + (-1)^{n/2} \frac{k^n - 1}{k^{n-1}} \frac{B_{n/2}}{n}$ where n is even

$= 0$ where n is odd AE. 9

$= (-1)^{n/2} \frac{k^n - 1}{k^{n-1}} \frac{B_{n/2}}{n}$ where n is even

$= (-1)^n \left\{1 + \frac{1}{2^{2n-1}} + \frac{1}{3^{2n-1}} - \frac{1}{6^{2n-1}}\right\} \frac{B_n}{4n}$ AE. 43

$= (-1)^n \left\{2 + \frac{1}{2^{2n-1}} - \frac{1}{4^{2n-1}}\right\} \frac{B_n}{4n}$ AE. 31

$= (-1)^n \frac{3^{2n} - 1}{3^{2n-1}} \frac{B_n}{4n}$ AE. 36

$= (-1)^n \frac{2^{2n} - 1}{2^{2n}} \frac{B_n}{n}$ AE. 26

$= \frac{1}{2}\left(1 + \frac{1}{3^{2n}}\right) B_{2n+1}\left(\frac{1}{4}\right) + \frac{1}{2^{2n+1}} B_{2n+1}\left(\frac{1}{6}\right)$ AE. 74

$= \frac{1}{2^{2n}} B_{2n+1}\left(\frac{1}{4}\right)$ AE. 63

Series No.

$$B_{2n+1}\left(\frac{1}{6}\right)$$

$$6^{2n+1}B_{2n+1}\left(\frac{1}{6}\right) - 3^{2n+1}B_{2n+1}\left(\frac{1}{3}\right)$$

$$B_{2n+1}\left(\frac{1}{4}\right)$$

$$B_{2n+1}\left(\frac{1}{3}\right)$$

$$B_{2n+1}\left(\frac{1}{2}\right)$$

$$B_{2n+1}\left(\frac{3}{4}\right)$$

$$A_{2n}\left(\frac{1}{12}\right) + A_{2n}\left(\frac{5}{12}\right)$$

$$A_{2n}\left(\frac{1}{8}\right) + A_{2n}\left(\frac{3}{8}\right)$$

$$A_{2n}\left(\frac{1}{6}\right)$$

$$A_{2n}\left(\frac{1}{4}\right)$$

$$A_{2n}\left(\frac{1}{3}\right)$$

$$A_{2n}\left(\frac{1}{2}\right)$$

$$(2n + 1)4^{2n+1}B_{2n+1}\left(\frac{1}{4}\right)$$

Reference

$$= \left(1 + \frac{1}{2^{2n}}\right) B_{2n+1}\left(\frac{1}{3}\right) \qquad \text{AE.52}$$

$$= (-1)^{n+1} H_n \qquad \text{AE. 51}$$

$$= A_{2n+1}\left(\frac{1}{4}\right) = (-1)^{n+1} \frac{E_n^*}{4^{2n+1}} \qquad \text{AE. 31}$$

$$= A_{2n+1}\left(\frac{1}{3}\right) = \frac{(-1)^{n+1}}{3^{2n+1}} I_n \qquad \text{AE. 35}$$

$$= A_{2n+1}\left(\frac{1}{2}\right) = 0 \qquad \text{AE. 26}$$

$$= -\, B_{2n+1}\left(\frac{1}{4}\right) \qquad \text{AE. 63}$$

$$= \frac{1}{2^{2n-1}} A_n\left(\frac{1}{6}\right) \qquad \text{AE. 77}$$

$$= \frac{1}{2^{2n-1}} A_{2n}\left(\frac{1}{4}\right) \qquad \text{AE. 66}$$

$$= (-1)^n \left\{\frac{1}{2^{2n-1}} - \frac{1}{6^{2n-1}}\right\} \frac{B_n}{4n} - A_{2n}\left(\frac{1}{3}\right) \qquad \text{AE. 42}$$

$$= (-1)^n \left\{\frac{1}{2^{2n-1}} - \frac{1}{4^{2n-1}}\right\} \frac{B_n}{4n} \qquad \text{AE. 31}$$

$$= (-1)^n \left\{1 - \frac{1}{3^{2n-1}}\right\} \frac{B_n}{4n} \qquad \text{AE. 36}$$

$$= (-1)^n \frac{2^{2n-1} - 1}{2^{2n}} \frac{B_n}{n} \qquad \text{AE. 26}$$

$$= -1 + (2n+1)_2 4^2\left(1 - \frac{1}{2}\right) B_1 \qquad \text{AE. 57}$$

$$- (2n+1)_4 4^4\left(1 - \frac{1}{2^3}\right) B_2$$

$$+ \ldots + (-1)^{n-1}(2n+1)4^{2n}\left(1 - \frac{1}{2^{2n-1}}\right) B_n$$

Series No.

$$(2n+1)6^{2n+1}B_{2n+1}\left(\frac{1}{3}\right)$$

The derivation of some of these numbers is from such series as No. (576) and No. (577) by putting $\theta = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$, etc.; but the original article in AE. should be consulted for a full description of the derivation.

(1143) $\frac{2}{3}\left\{I_0 - \frac{I_1}{2!}a^2 + \frac{I_2}{4!}a^4 - \ldots \infty\right\}$

(1144) $\frac{3}{2}I_n - (2n)_2 I_{n-1} + (2n)_4 I_{n-2} + \ldots + (-1)^{n-1}(2n)_2 I_1 + (-1)^n I_0$

(1145) $E_n^* - (2n)_2 E_{n-1}^* + (2n)_4 E_{n-2}^* + \ldots + (-1)^{n-1}(2n)_2 E_1^* + (-1)^n E_0^*$

In No. (1142), see No. (1130) for values of I, E^*, and H. n_r is the binomial coefficient

$$\frac{n(n-1)\ldots(n-r+1)}{r!}$$

(1146) $2^2 B_2\left(\frac{1}{2}\right) = -\frac{1}{2}$

$2^4 B_4\left(\frac{1}{2}\right) = \frac{1}{4}$

$2^6 B_6\left(\frac{1}{2}\right) = -\frac{1}{2}$

$2^8 B_8\left(\frac{1}{2}\right) = \frac{51}{24}$

Reference

$$= -1 + (2n+1)_2 6^2\left(1 - \frac{1}{2}\right) B_1$$ AE. 51

$$- (2n+1)_4 6^4\left(1 - \frac{1}{2^3}\right) B_2$$

$$+ \ldots + (-1)^{n-1}(2n+1)6^{2n}\left(1 - \frac{1}{2^{2n-1}}\right) B_n$$

$$= \frac{1}{1 + e^a + e^{-a}}$$ AE. 37

$$= 0$$ AE. 37

$$= 0$$ AE. 37

$$2^2 A_2\left(\frac{1}{2}\right) = -\frac{1}{6}$$ AE. 26

$$2^4 A_4\left(\frac{1}{2}\right) = \frac{7}{60}$$

$$2^6 A_6\left(\frac{1}{2}\right) = -\frac{31}{126}$$

$$2^8 A_8\left(\frac{1}{2}\right) = \frac{127}{120}$$

Series No.

$$3^2B_2\left(\frac{1}{3}\right) = -1$$

$$3^4B_4\left(\frac{1}{3}\right) = 1$$

$$3^6B_6\left(\frac{1}{3}\right) = -\frac{13}{3}$$

$$3^8B_8\left(\frac{1}{3}\right) = 41$$

$$4^2B_2\left(\frac{1}{4}\right) = -\frac{3}{2}$$

$$4^4B_4\left(\frac{1}{4}\right) = \frac{9}{4}$$

$$4^6B_6\left(\frac{1}{4}\right) = -\frac{33}{2}$$

$$4^8B_8\left(\frac{1}{4}\right) = \frac{6579}{24}$$

$$B_3\left(\frac{1}{8}\right) = \frac{7}{8^3}$$

$$B_5\left(\frac{1}{8}\right) = -\frac{119}{8^5}$$

Reference

$$3^2 A_2\left(\frac{1}{3}\right) = -\frac{1}{4}$$ AE. 36

$$3^4 A_4\left(\frac{1}{3}\right) = \frac{13}{40}$$

$$3^6 A_6\left(\frac{1}{3}\right) = -\frac{121}{84}$$

$$3^8 A_8\left(\frac{1}{3}\right) = \frac{1093}{80}$$

$$4^2 A_2\left(\frac{1}{4}\right) = -\frac{1}{6}$$ AE. 31

$$4^4 A_4\left(\frac{1}{4}\right) = \frac{7}{60}$$

$$4^6 A_6\left(\frac{1}{4}\right) = -\frac{31}{126}$$

$$4^8 A_8\left(\frac{1}{4}\right) = \frac{127}{120}$$

AE. 61

AE. 61

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