Appendix A

Formulae and Summary

Note to student: It is not useful to memorise all the formulae, partly because many of the complicated formulae may be obtained from the simpler ones. Rather, you should familiarise yourself with the simpler formulae by solving a number of practice questions which use those. Consult your school to check which formulae you are expected to know by heart for examinations, and which others will be provided to you in a help sheet.

Exponents and Logarithms

- For $a, b > 0$,

$$a^{-1} = \frac{1}{a}\,. \tag{A.1}$$

$$a^0 = 1\,. \tag{A.2}$$

$$a^{xy} = (a^x)^y\,. \tag{A.3}$$

$$a^x a^y = a^{x+y}\,. \tag{A.4}$$

$$(ab)^x = a^x b^x\,. \tag{A.5}$$

$$\sqrt{ab} = \sqrt{a}\,\sqrt{b}. \tag{A.6}$$

- For $a > 1$, $y = a^x$ is positive and an increasing function of x along the real line; for $0 < a < 1$, a^x is a decreasing function of x.

- The basic relation between exponents and logarithms is

$$y = b^x \Leftrightarrow x = \log_b y \ . \tag{A.7}$$

- For $a, b > 0,\ a \neq 1,\ b \neq 1$ and $P, Q > 0$,

$$\log_b 1 = 0. \tag{A.8}$$
$$\log_b PQ = \log_b P + \log_b Q\,. \tag{A.9}$$
$$\log_b \frac{P}{Q} = \log_b P - \log_b Q\,. \tag{A.10}$$
$$\log_b P^c = c \log_b P\,. \tag{A.11}$$
$$\log_b P = \frac{\log_a P}{\log_a b}\,. \tag{A.12}$$
$$\log_b a = \frac{1}{\log_a b}\,. \tag{A.13}$$

- For $a > 1$ and $x > 0$, $y = \log_a x$ is an increasing function of x; it is positive for $x > 1$, negative for $x < 1$ and vanishes at $x = 1$. The graph of $y = \log_a x$ may be obtained by reflecting the exponential curve $y = a^x$ about the line $y = x$.

Binomial Expansions

For a positive integer n, the Binomial Theorem states

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + \binom{n}{n-1} a b^{n-1} + b^n \tag{A.14}$$

where the binomial coefficient is defined by

$${}^nC_r \equiv \binom{n}{r} = \frac{n!}{r!(n-r)!} \tag{A.15}$$

and $n! \equiv n \times (n-1) \times (n-2) \times \ldots. \times 2 \times 1$ with $0! \equiv 1$.

Note that the $(r+1)$-th term in the above expansion is given by $\binom{n}{r} a^{n-r} b^r$.

Polynomials

- **Remainder Theorem**: If the polynomial $P(x)$ is divided by $(x-\alpha)$, the remainder is $P(\alpha)$.
  In other words, $P(x) \equiv (x-\alpha)Q(x) + R$ with $P(\alpha) = R$.

- **Factorisation Theorem**:
  $P(\alpha) = 0$ if and only if $P(x) \equiv (x-\alpha)Q(x)$.

- For the quadratic equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}. \tag{A.16}$$

  where

$$\Delta \equiv b^2 - 4ac \tag{A.17}$$

  is called the **discriminant**. The two roots are real if and only if $\Delta \geq 0$; the case $\Delta = 0$ corresponds to a repeated root. Denoting the two roots of a quadratic equation by α and β, then

$$\begin{aligned} \alpha + \beta &= \frac{-b}{a} \,. \\ \alpha\beta &= \frac{c}{a} \,. \end{aligned} \tag{A.18}$$

  Other common relations are

$$\begin{aligned} \alpha - \beta &= \pm\sqrt{(\alpha-\beta)^2} \\ &= \pm\sqrt{(\alpha+\beta)^2 - 4\alpha\beta} \,. \end{aligned} \tag{A.19}$$

  and

$$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta \,. \tag{A.20}$$

- The shape of the curve $y = ax^2 + bx + c$ is determined by the sign of a: When $a > 0$, the curve has a minimum point while for $a < 0$ it has a maximum. For a cubic curve $y = ax^3 + bx^2 + cx + d$, the sign of the leading coefficient again determines its the main shape. If $a > 0$, the curve rises upwards for large positive x and decreases for large negative x. In between, it will either have a local minimum and a local maximum or gently undulate.

Partial Fractions

- Given a rational function $P(x)/Q(x)$, its decomposition into partial fractions is as follows: First use long division to reduce the degree of the numerator to below that of the denominator. Next, each factor of $(x-a)$ in $Q(x)$ would require a partial fraction $A/(x-a)$.
- If the factor is repeated in Q, for example $(x-a)^2$, then one uses two partial fractions $A_1/(x-a)$ and $A_2/(x-a)^2$ for that factor.
- If Q contains a term that cannot be factorised (using real numbers), for example x^2+x+1, then the partial fraction for that term is of the form $(Ax+B)/(x^2+x+1)$, that is, the numerator is one degree lower than the denominator.

Simultaneous Equations

- A $m \times n$ matrix A multiplying from the left a $n \times p$ matrix B yields a $m \times p$ matrix C; that is $C = AB$. The c_{ij} element of C is obtained by multiplying the i-th row of A to the j-th column of B term by term and adding the pieces.
- The pair of simultaneous equations in the variables (x, y):

$$ax + by = e \tag{A.21}$$
$$cx + dy = f\,, \tag{A.22}$$

may be written as the matrix equation $MX = R$ with

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{A.23}$$

$$X = \begin{pmatrix} x \\ y \end{pmatrix} \tag{A.24}$$

and

$$R = \begin{pmatrix} e \\ f \end{pmatrix}. \tag{A.25}$$

- The **determinant** of the 2×2 matrix M is defined by det(M)$=$ $ad - bc$. If det(M)$\neq 0$ then one may define an **inverse matrix**

$$M^{-1} = \frac{1}{\text{det(M)}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{A.26}$$

and the solution to the problem is

$$X = M^{-1}R = \frac{1}{\text{det(M)}} \begin{pmatrix} de - bf \\ -ce + af \end{pmatrix} . \tag{A.27}$$

- If det(M)$= 0$ the matrix is termed **singular** and there is no inverse matrix. The inverse matrix satisfies

$$MM^{-1} = M^{-1}M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \tag{A.28}$$

the last matrix being the **identity matrix**, usually denoted by the letter I.

Trigonometry

- For angle measurements in **radian**, $\theta \equiv s/R$ where s is the arc length of circle subtended by that angle. Therefore 2π radians equals $360°$.

- For a right-angled triangle ABC, with $C = 90°$, $\sin A = a/c$, $\cos A = b/c$ and $\tan A = \sin A/ \cos A = a/b$, where the small case letters denote lengths opposite the corresponding angles.

- The sin and cos functions range over the interval $[-1, 1]$ while the tan functions ranges over the real line.

- Periodicity: $\sin(\theta + 360°) = \sin(\theta)$, $\cos(\theta + 360°) = \cos(\theta)$ and $\tan(\theta + 180°) = \tan(\theta)$.

- The **principal values** for the inverse functions $\sin^{-1}$ and $\tan^{-1}$ are those that lie in the range $-\pi/2 \leq y \leq \pi/2$ while for the $\cos^{-1}$ function has the range $0 \leq y \leq \pi$.

- The functions csc, sec and cot are reciprocals of the sin, cos and tan functions ($\csc x$ is also written as cosec x).
- Phenomena that are described by an equation of the form $y(x) = A + B\sin(kx + C)$ are called **sinusoidal**.
- Some identities:

$$\sin^2 A + \cos^2 A = 1 \; . \tag{A.29}$$
$$1 + \tan^2 A = \sec^2 A \; . \tag{A.30}$$
$$\sin(-A) = -\sin A \; . \tag{A.31}$$
$$\cos(-A) = \cos A \; . \tag{A.32}$$
$$\tan(-A) = -\tan A \; . \tag{A.33}$$

- Addition Formulae:

$$\sin(A \pm B) = \sin A \;\cos B \pm \cos A \;\sin B \; . \tag{A.34}$$
$$\cos(C \pm D) = \cos C \;\cos D \mp \sin C \;\sin D \; . \tag{A.35}$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \;\tan B} \; . \tag{A.36}$$

- Special Cases of addition formulae:

$$\sin(90^\circ \pm A) = \cos A \; . \tag{A.37}$$
$$\sin(180^\circ \pm A) = \mp \sin A \; . \tag{A.38}$$
$$\cos(90^\circ \pm A) = \mp \sin A \; . \tag{A.39}$$
$$\cos(180^\circ \pm A) = -\cos A \; . \tag{A.40}$$
$$\tan(90^\circ \pm A) = \mp \cot A. \tag{A.41}$$

- Double Angle Formulae:

$$\sin 2A = 2\sin A \cos A \; . \tag{A.42}$$
$$\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \cos^2 A - \sin^2 A \; . \tag{A.43}$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A} \; . \tag{A.44}$$

- Factor Formulae:

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} . \quad (A.45)$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2} . \quad (A.46)$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} . \quad (A.47)$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2} . \quad (A.48)$$

- R-Formulae:

$$a\cos\theta \pm b\sin\theta = R\cos(\theta \mp \alpha) . \quad (A.49)$$

$$a\sin\theta \pm b\cos\theta = R\sin(\theta \pm \alpha) . \quad (A.50)$$

with $R = \sqrt{a^2+b^2}$, $\tan\alpha = b/a$.

Properties of Triangles

- For a right-angled triangle, a mnemonic is "**soh-cah-toa**", which summarises
  $\sin = Oppositeside/Hypotenuse$,
  $\cos = Adjacentside/Hypotenuse$ and
  $\tan = Opposite/Adjacent$.

- Area of a triangle:

$$A = \frac{1}{2}ab\ \sin C . \quad (A.51)$$

- The **sine rule**:

$$\frac{c}{\sin C} = \frac{b}{\sin B} = \frac{a}{\sin A} . \quad (A.52)$$

  Each ratio in the formula equals the diameter of a circle which circumscribes the triangle (the centre of the circle being at the intersection of the perpendicular bisectors of the three sides).

- The **cosine rule**:

$$c^2 = a^2 + b^2 - 2ab\ \cos C . \quad (A.53)$$

Elevation, Depression and Bearing

- If a point P is above the horizontal through the observation point O, then the angle that OP makes with the horizontal is the **angle of elevation** of P. Similarly if P were below the horizontal then the corresponding angle would be termed **angle of depression**.

- A **bearing** denotes a direction relative to North. It is usually expressed by a clockwise angle measured in degrees. For example, 065° would refer to the direction 65° clockwise from North. Such bearings are called "absolute bearings".

- It is also convenient to use "relative bearings", which define an angle relative to a chosen axis, for example the axis along which an aircraft is pointing.

Coordinate Geometry

- If (x, y) and (x_1, y_1) are two points on the line,

$$\frac{y - y_1}{x - x_1} = m = \tan\alpha\,, \qquad \text{(A.54)}$$

  where $0 \leq \alpha < \pi$ is the angle the line makes with the positive x-axis. The equation for a straight line may be written in the form

$$y = mx + c \qquad \text{(A.55)}$$

  where m is the slope and c the intercept on the y-axis.

- If another line is perpendicular to (A.55), its slope must be $-1/m$.

- The mid-point of a line joining two points (x_1, y_1) and (x_2, y_2) is given by $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

- The equation for a circle is

$$(x - x_0)^2 + (y - y_0)^2 = r^2\,, \qquad \text{(A.56)}$$

where (x_0, y_0) is the centre and r the radius. Given the form

$$x^2 + y^2 - 2ax - 2by + c = 0\,, \tag{A.57}$$

for some constants a, b, c, you can complete the squares to get $(x-a)^2+(y-b)^2 = a^2+b^2-c$; if $c < a^2+b^2$ then (5.4) represents a circle with centre (a, b) and radius $R = \sqrt{a^2 + b^2 - c}$.

- A tangent to a circle at any point P is perpendicular to the line OP where O is the centre of the circle. Therefore the **normal** to the circle at P lies along OP.

- Given three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, with their relative order being anti-clockwise, the area of the triangle ABC may be obtained from the formula

$$A = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix} \tag{A.58}$$

where the terms are generated as follows. First, start at the left of the top row and multiply each term in the top row by a term one step to the right in the bottom row, adding the pieces: $x_1y_2 + x_2y_3 + x_3y_1$. Then start at the right of the top row and multiply each term in the top row by a term one step to the left in the bottom row, adding the pieces: $x_1y_3+x_3y_2+x_2y_1$. Finally, subtract the two contributions and include the overall 1/2 to get

$$A = \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_3y_2 - x_2y_1)\,. \tag{A.59}$$

- The area of a polygon may be determined by dividing the polygon into triangles and using the above formula for each triangle.

Plane Geometry

- A **cyclic polygon** is one whose vertices lie on the circumference of a circle. Every triangle is in fact cyclic.

- **Inscribed Angle Theorem**
  Let A and B be two points on the circumference of a circle. The angle subtended by A and B at the centre of the circle is twice that subtended at a point C on the circumference.

- **Tangent-Chord Theorem (Alternate Segment Theorem)**
  Let the triangle ABC be inscribed in a circle and a tangent drawn at point A. Let D be another point on the tangent line such that D and C are on opposite sides of the line AB. Then $\angle DAB = \angle BCA$.

- **Intersecting Chords Theorem**
  Let A, B, C, and D be points on the circumference of a circle and X the point of intersection of the lines AC and BD. Then the triangle ABX is similar to triangle DCX.

- **Tangent-Secant Theorem**
  Let A, B and C be points on the circumference of a circle and let the tangent at A meet the line CB produced at D. Then $(DA)^2 = DB \times DC$.

Differential Calculus

- Let f and g be functions of x, and A, B, n represent constants in the following.

$$\frac{d}{dx}Ax^n = Anx^{n-1}\,. \tag{A.60}$$

$$\frac{d}{dx}e^x = e^x\,. \tag{A.61}$$

$$\frac{d\ln x}{dx} = \frac{1}{x}\,. \tag{A.62}$$

$$\frac{d}{dx}\sin x = \cos x\,. \tag{A.63}$$

$$\frac{d}{dx}\cos x = -\sin x\,. \tag{A.64}$$

$$\frac{d}{dx}\tan x = \sec^2 x\,. \tag{A.65}$$

$$\begin{aligned}
\frac{d}{dx}(f+g) &= \frac{df}{dx} + \frac{dg}{dx}\,. && \text{(A.66)}\\
\frac{d(fg)}{dx} &= f\frac{dg}{dx} + g\frac{df}{dx}\,. \quad \textbf{Product Rule} && \text{(A.67)}\\
\frac{d}{dx}\frac{f}{g} &= \frac{gf' - fg'}{g^2}\,. \quad \textbf{Quotient Rule} && \text{(A.68)}\\
\frac{df(g(x))}{dx} &= \frac{df}{dg} \times \frac{dg}{dx}\,. \quad \textbf{Chain Rule} && \text{(A.69)}\\
\frac{df}{dx} &= 1/\left(\frac{dx}{df}\right)\,. && \text{(A.70)}
\end{aligned}$$

- The **stationary** points of $y = f(x)$ are those for which $\frac{df}{dx} = 0$. The stationary points are **local minima, local maxima, or points of inflexion**. If $\frac{d^2f}{dx^2} >\ (<)\ 0$ at the stationary point, it is a local minimum (maximum). Stationary points with $\frac{d^2f}{dx^2} = 0$ can be examined using the "first derivative test". Local extrema need not be global extrema.

- If Δx is small but not strictly zero, we may form the approximation
$$\Delta y \approx \left(\frac{dy}{dx}\right) \times \Delta x\,. \tag{A.71}$$

Integral Calculus

- **Fundamental Theorem of Calculus**:
$$\int_a^b f(x)dx = F(b) - F(a)\,, \tag{A.72}$$
where $F(x)$ is a function that satisfies
$$\frac{dF(x)}{dx} = f(x)\ . \tag{A.73}$$
Hence
$$\int_a^b \frac{dF(x)}{dx}\, dx = F(b) - F(a)\,. \tag{A.74}$$

- Indefinite integrals:

$$\int f(x)dx = F(x) + C\,, \qquad \text{(A.75)}$$

where (A.73) still holds and C is a constant of integration which can be fixed once we have more information about the problem.

- In the formulae for indefinite integrals below, f and g are functions of x while a, b, n are constants; C is a constant of integration.

$$\int (a+bx)^n \; dx = \frac{(a+bx)^{n+1}}{b(n+1)} + C, \quad n \neq -1\,. \qquad \text{(A.76)}$$

$$\int \frac{1}{a+bx} \; dx = \frac{1}{b}\ln(a+bx) + C\,. \qquad \text{(A.77)}$$

$$\int \sin(a+bx) \; dx = \frac{-1}{b}\cos(a+bx) + C\,. \qquad \text{(A.78)}$$

$$\int \cos(a+bx) \; dx = \frac{1}{b}\sin(a+bx) + C\,. \qquad \text{(A.79)}$$

$$\int e^{(a+bx)} \; dx = \frac{1}{b}e^{(a+bx)} + C\,. \qquad \text{(A.80)}$$

$$\int (f+g) \; dx = \int f \; dx + \int g \; dx\,. \qquad \text{(A.81)}$$

- The following identity holds:

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx\,. \qquad \text{(A.82)}$$

- For calculating areas between the curve $y = f(x)$ and the x-axis, note that if $f(x) < 0$ within a region $x_1 \leq x \leq x_2$, the integral in that region would give a negative value and the area is then the negative of the integral.

- One may also evaluate the area between a curve and the y-axis. In this case the integral would be $\int_{y_1}^{y_2} x \; dy$.