# 高等数学公式汇总

## 第一章　一元函数的极限与连续

### 1、一些初等函数公式：

和差角公式：

$$\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$$
$$\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$$
$$\tan(\alpha\pm\beta)=\frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\cdot\tan\beta}$$
$$\cot(\alpha\pm\beta)=\frac{\cot\alpha\cdot\cot\beta\mp1}{\cot\beta\pm\cot\alpha}$$
$$sh(\alpha\pm\beta)=sh\alpha ch\beta\pm ch\alpha sh\beta$$
$$ch(\alpha\pm\beta)=ch\alpha ch\beta\pm sh\alpha sh\beta$$

和差化积公式：

$$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$\sin\alpha-\sin\beta=2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$
$$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$\cos\alpha-\cos\beta=2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

积化和差公式：

$$\sin\alpha\cos\beta=\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$
$$\cos\alpha\sin\beta=\frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$$
$$\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$$
$$\sin\alpha\sin\beta=\frac{1}{2}[\cos(\alpha+\beta)-\cos(\alpha-\beta)]$$

倍角公式：

$$\sin2\alpha=2\sin\alpha\cos\alpha$$
$$\cos2\alpha=2\cos^2\alpha-1$$
$$=1-2\sin^2\alpha=\cos^2\alpha-\sin^2\alpha$$
$$\tan2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha}$$
$$\cot2\alpha=\frac{\cot^2\alpha-1}{2\cot\alpha}$$
$$sh2\alpha=2sh\alpha ch\alpha$$
$$ch2\alpha=1+2sh^2\alpha=$$
$$=2ch^2\alpha-1=ch^2\alpha+sh^2\alpha$$

$\sin^2\alpha+\cos^2\alpha=1;\tan^2 x+1=\sec^2 x;$

$\cot^2 x+1=\csc^2 x;ch^2x-sh^2x=1$

半角公式：

$$\sin\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$\cos\frac{\alpha}{2}=\pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\tan\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}=\frac{1-\cos\alpha}{\sin\alpha}=\frac{\sin\alpha}{1+\cos\alpha}$$

$$\cot\frac{\alpha}{2}=\pm\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}=\frac{1+\cos\alpha}{\sin\alpha}=\frac{\sin\alpha}{1-\cos\alpha}$$

双曲正弦$:shx=\dfrac{e^x-e^{-x}}{2}$；反双曲正弦$:arshx=\ln(x+\sqrt{x^2+1})$

双曲余弦$:chx=\dfrac{e^x+e^{-x}}{2}$；反双曲余弦$:archx=\pm\ln(x+\sqrt{x^2-1})$

双曲正切$:thx=\dfrac{shx}{chx}=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$；反双曲正切$:arthx=\dfrac{1}{2}\ln\dfrac{1+x}{1-x}$

$(a^3\pm b^3)=(a\pm b)(a^2\mp ab+b^2)$，$1^2+2^2+\cdots+n^2=\dfrac{n(n+1)(2n+1)}{6}$

$1^3+2^3+\cdots+n^3=\dfrac{n^2(n+1)^2}{4}$

2、极限

- 常用极限：$|q|<1,\lim\limits_{n\to\infty}q^n=0$；$a>1,\lim\limits_{n\to\infty}\sqrt[n]{a}=1$；$\lim\limits_{n\to\infty}\sqrt[n]{n}=1$

- 若$f(x)\to 0,g(x)\to\infty,$则 $\lim[1\pm f(x)]^{g(x)}=e^{\lim\frac{\ln(1+f(x))}{1/g(x)}}\xrightarrow{\ln(1+f(x))\sim f(x)}e^{\pm\lim[f(x)g(x)]}$

- 两个重要极限

$$\lim_{x\to 0}\frac{\sin x}{x}=1,\lim_{x\to\infty}\frac{\sin x}{x}=0;\lim_{x\to\infty}(1+\frac{1}{x})^x=e=\lim_{x\to 0}(1+x)^{\frac{1}{x}}$$

- 常用等价无穷小:

$$1-\cos x\sim\frac{1}{2}x^2;\ x\sim\sin x\sim\arcsin x\sim\arctan x;\sqrt[n]{1+x}-1\sim\frac{1}{n}x;$$

$$a^x-1\sim x\ln a;\ e^x\sim x+1;(1+x)^a\sim 1+ax;\ \ln(1+x)\sim x$$

3、连续：

定义：$\lim\limits_{\Delta x \to 0} \Delta y = 0; \lim\limits_{x \to x_0} f(x) = f(x_0)$

极限存在$\Leftrightarrow \lim\limits_{x \to x_0^-} f(x) = \lim\limits_{x \to x_0^+} f(x)$或$f(x_0^-) = f(x_0^+)$

## 第二章　导数与微分

1、基本导数公式：

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \tan\alpha$$

导数存在$\Leftrightarrow f'_-(x_0^-) = f'_+(x_0^+)$

$C' = 0;\ (x^a)' = ax^{a-1};\ (\sin x)' = \cos x;\ (\cos x)' = \sin x;\ (\tan x)' = \sec^2 x;\ (\cot x)' = -\csc^2 x;$

$(\sec x)' = \sec x \cdot \tan x;\ (\csc x)' = -\csc x \cdot ctgx;\ (a^x)' = a^x \ln a; (e^x)' = e^x;$

$(\log_a |x|)' = \dfrac{1}{x \ln a};\ (\ln|x|)' = \dfrac{1}{x};\ (\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}};\ (\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}};$

$(\arctan x)' = \dfrac{1}{1+x^2};\ (arc\cot x)' = -\dfrac{1}{1+x^2};\ (shx)' = hx; (chx)' = shx;$

$(thx)' = \dfrac{1}{ch^2 x};\ (arshx)' = \dfrac{1}{\sqrt{1+x^2}};\ (archx)' = \dfrac{1}{\sqrt{x^2-1}}; (arthx)' = \dfrac{1}{x^2-1}$

2、高阶导数：

$$(x^n)^{(k)} = \frac{n!}{(n-k)!} x^{n-k} \Rightarrow (x^n)^{(n)} = n!;\ (a^x)^{(n)} = a^x \ln^n a \Rightarrow (e^x)^{(n)} = e^x$$

$$(\frac{1}{x})^{(n)} = \frac{(-1)^n n!}{x^{n+1}};\ (\frac{1}{x+a})^{(n)} = \frac{(-1)^n n!}{(x+a)^{n+1}};\ (\frac{1}{a-x})^{(n)} = \frac{n!}{(a-x)^{n+1}}$$

$$(\sin kx)^{(n)} = k^n \cdot \sin(kx + n \cdot \frac{\pi}{2});\ (\cos kx)^{(n)} = k^n \cdot \cos(kx + n \cdot \frac{\pi}{2});$$

$$[\ln(a+x)]^{(n)} = (-1)^{n-1} \frac{(n-1)!}{(a+x)^n} \Rightarrow [\ln(x)]^{(n)} = (\frac{1}{x})^{(n-1)} = (-1)^{n-1} \frac{(n-1)!}{x^n}$$

✧ 牛顿-莱布尼兹公式：

$$(uv)^{(n)} = \sum_{k=0}^{n} C_n^k u^{(n-k)} v^{(k)}$$

$$= u^{(n)} v + n u^{(n-1)} v' + \frac{n(n-1)}{2!} u^{(n-2)} v'' + \cdots + \frac{n(n-1)\cdots(n-k+1)}{k!} u^{(n-k)} v^{(k)} + \cdots + u v^{(n)}$$

3、微分：

$\Delta y = f(x+\Delta x) - f(x) = dy + o(\Delta x);\ dy = f'(x_0)\Delta x = f'(x)dx;$

连续$\Rightarrow$极限存在$\Leftrightarrow$收敛$\Rightarrow$有界；　　　　可微$\Leftrightarrow$可导$\Leftrightarrow$左导=右导$\Rightarrow$连续；

不连续$\Rightarrow$不可导

## 第三章　　微分中值定理与微分的应用

1、基本定理

拉格朗日中值定理：$f(b)-f(a)=f'(\xi)(b-a),\xi\in(a,b)$

柯西中值定理：$\dfrac{f(b)-f(a)}{F(b)-F(a)}=\dfrac{f'(\xi)}{F'(\xi)},\xi\in(a,b)$

当$F(x)=x$时，柯西中值定理就是拉格朗日中值定理。

2、

泰勒公式：$f(x)=f(x_0)+f'(x_0)(x-x_0)+\dfrac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\dfrac{f^{(n)}(x_0)}{n!}(x-x_0)^n+R_n(x)$

余项：$R_n(x)=\begin{cases} o((x-x_0)^n) \\ \dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}=\dfrac{f^{(n+1)}(x_0+\theta(x-x_0))}{(n+1)!}(x-x_0)^{n+1} \end{cases};\ (\xi\in(x_0,x),\theta\in(0,1))$

麦克劳林公式：$f(x)=f(0)+f'(0)(x)+\dfrac{f''(0)}{2!}(x)^2+\cdots+\dfrac{f^{(n)}(0)}{n!}(x)^n+\dfrac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1};\ (\theta\in(0,1))$

✧　常用初等函数的展式：

$e^x=1+x+\dfrac{x^2}{2!}+\cdots+\dfrac{x^n}{n!}+R_n(x);R_n(x)=\dfrac{e^{\theta x}}{(n+1)!}x^{n+1};(\theta\in(0,1))$

$\sin x=x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\cdots+(-1)^{m-1}\dfrac{x^{2m-1}}{(2m-1)!}+R_{2m}(x);R_{2m}(x)=\dfrac{\sin[\theta x+(2m+1)\frac{\pi}{2}]}{(2m+1)!}x^{2m+1};(\theta\in(0,1))$

$\cos x=1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\cdots+(-1)^m\dfrac{x^{2m}}{(2m)!}+R_{2m+1}(x);R_{2m+1}(x)=\dfrac{\cos[\theta x+(m+1)\pi]}{(2m+2)!}x^{2m+2};(\theta\in(0,1))$

$$\ln(1+x)=x-\frac{x^2}{2!}+\frac{x^4}{3!}-\cdots+(-1)^{n-1}\frac{x^n}{n}+R_n(x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1};$$

$$R_n(x)=\frac{(-1)^n}{(n+1)(1+\theta x)^{n+1}}x^{n+1};(\theta\in(0,1))$$

$$(1+x)^{\alpha}=1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\cdots+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n+R_n(x);$$

$$R_n(x)=\frac{\alpha(\alpha-1)\cdots(\alpha-n)}{(n+1)!}(1+\theta x)^{\alpha-n-1}x^{n+1};(\theta\in(0,1))$$

$$\Rightarrow\frac{1}{1+x}=\ln'(1+x)=1-x+x^2+\cdots+(-1)^n x^n=\sum_{n=0}^{\infty}(-1)^n x$$

3、

弧微分公式：$ds=\sqrt{1+y'^2}dx=\sqrt{x'(t)+y'(t)^2}dt=\sqrt{\rho^2+\rho'^2}d\theta$

平均曲率：$\overline{K}=\left|\frac{\Delta\alpha}{\Delta s}\right|$.($\Delta\alpha$：从M点到M′点，切线斜率的倾角变化量；$\Delta$s:$MM'$弧长)

M点的曲率：$K=\lim\limits_{\Delta s\to 0}\left|\frac{\Delta\alpha}{\Delta s}\right|=\left|\frac{d\alpha}{ds}\right|=\frac{|y''|}{\sqrt{(1+y'^2)^3}}=\frac{|\varphi'(t)\psi''(t)-\varphi''(t)\psi'(t)|}{[\varphi'^2(t)+\psi'^2(t)]^{\frac{3}{2}}}$.

直线的曲率：$K=0$; 半径为$R$的圆的曲率：$K=\frac{1}{R}$.

曲线在点$M$处的曲率半径：$\rho=\frac{1}{K}=\frac{\sqrt{(1+y'^2)^3}}{|y''|}$

## 第四章　不定积分

1、常用不定积分公式：

$\int f(x)dx=F(x)+C$; $(\int f(x)dx)'=f(x)$; $\int F'(x)dx=F(x)+C$

$\int x^{\mu}dx=\frac{x^{\mu+1}}{\mu+1}+C(\mu\neq-1)$; $\int\frac{1}{x}dx=\ln|x|+C$;

$\int a^x dx=\frac{a^x}{\ln a}+C$; $\int e^x dx=e^x+C$;

$$\int \sin x dx = -\cos x + C; \quad \int \cos x dx = \sin x + C;$$

$$\int \tan x dx = -\ln|\cos x| + C; \quad \int \cot x dx = \ln|\sin x| + C;$$

$$\int \sec x dx = \ln|\sec x + \tan x| + C;$$

$$\int \csc x dx = \ln|\csc x - \cot x| + C = \ln\left|\tan\frac{x}{2}\right| + C = -\ln|\csc x + \cot x| + C;$$

$$\int \sec^2 x dx = \int \frac{dx}{\cos^2 x} = \tan x + C; \quad \int \csc^2 x dx = \int \frac{dx}{\sin^2 x} = -\cot x + C;$$

$$\int \sec x \cdot \tan x dx = \sec x + C; \quad \int \csc x \cdot \cot x dx = -\csc x + C;$$

$$\int shx dx = chx + C; \quad \int chx dx = shx + C;$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C = -\arccos x + C; \quad \int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C;$$

$$\int \frac{dx}{1+x^2} = \arctan x + C = -\mathrm{arc}\cot x + C; \quad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a} + C;$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C; \quad \int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\frac{a+x}{a-x} + C;$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln(x + \sqrt{x^2 \pm a^2}) + C;$$

$$\int \sqrt{x^2 \pm a^2} dx = \frac{x}{2}\sqrt{x^2 \pm a^2} + \frac{a^2}{2}\ln(x + \sqrt{x^2 \pm a^2}) + C;$$

$$\int \sqrt{a^2 - x^2} dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C$$

2、常用凑微分公式：

$$\frac{dx}{\sqrt{x}} = 2d\sqrt{x}; \quad \frac{dx}{x^2} = -d(\frac{1}{x}); \quad \frac{dx}{x} = d(\ln x);$$

$$\frac{xdx}{\sqrt{1+x^2}} = d(\sqrt{1+x^2}); \quad (1-\frac{1}{x^2})dx = d(x+\frac{1}{x})$$

$$\frac{dx}{\cos x \sin x} = d(\ln \tan x);$$

3、有特殊技巧的积分

(1) $\displaystyle\int \frac{dx}{a\sin x + b\cos x} = \frac{1}{\sqrt{a^2+b^2}}\int \frac{1}{\sin(x+\varphi)}dx$

(2) $\displaystyle\int \frac{c\sin x + d\cos x}{a\sin x + b\cos x}dx = Ax + B\ln|a\sin x + b\cos x| + C$

(3) $\displaystyle\int \frac{x^2+1}{x^4+1}dx = \int \frac{1}{(x-\frac{1}{x})^2 + (\sqrt{2})^2}d(x-\frac{1}{x})$

# 第五章　定积分

## 1、基本概念

$$\int_a^b f(x)dx=\lim_{\lambda\to0}\sum_{i=1}^{n}f(\xi_i)\Delta x_i=\lim_{n\to0}\sum_{i=1}^{n}f(\frac{i}{n})\frac{1}{n}=F(b)-F(a)=F(x)\Big|_a^b,\quad (F'(x)=f(x))$$

连续$\Rightarrow$可积;有界+有限个间断点$\Rightarrow$可积;

可积$\Rightarrow$有界; 连续$\Rightarrow$原函数存在

$$\Phi(x)=\int_a^x f(t)dt\Rightarrow\Phi'(x)=f(x)$$

$$\frac{d}{dx}\int_{\psi(x)}^{\varphi(x)}f(t)dt=f[\varphi(x)]\varphi'(x)-f[\psi(x)]\psi'(x)$$

$$\int_b^a f(x)dx=\int_\beta^\alpha f(\varphi(t))\varphi'(t)dt,\quad \int_b^a u(x)dv(x)=u(x)v(x)-\int_b^a v(x)du(x)$$

## 2、常用定积分公式：

$$\int_{-a}^{a}f(x)dx=\int_0^a[f(x)+f(-x)]dx;$$

$f(x)$为偶函数,$\int_{-a}^{a}f(x)dx=2\int_0^a f(x)dx$； $f(x)$为奇函数,$\int_{-a}^{a}f(x)dx=0$

$$\int_0^{\frac{\pi}{2}}f(\sin x)dx=\int_0^{\frac{\pi}{2}}f(\cos x)dx;\quad \int_0^{\frac{\pi}{2}}xf(\sin x)dx=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}f(\sin x)dx=\pi\int_0^{\frac{\pi}{2}}f(\sin x)dx$$

$$\int_a^{a+\mathrm{T}}f(x)dx=\int_0^{\mathrm{T}}f(x)dx=\int_{-\frac{\mathrm{T}}{2}}^{\frac{\mathrm{T}}{2}}f(x)dx;\quad \int_a^{a+n\mathrm{T}}f(x)dx=n\int_0^{\mathrm{T}}f(x)dx$$

Wallis 公式：

$$I_n=\int_0^{\frac{\pi}{2}}\sin^n xdx=\int_0^{\frac{\pi}{2}}\cos^n xdx=\frac{n-1}{n}I_{n-2}=\begin{cases}\frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\cdots\cdot\frac{n-3}{n-2}\cdot\frac{n-1}{n}, n\text{为正偶数}\\ \frac{2}{3}\cdot\frac{4}{5}\cdot\cdots\cdot\frac{n-3}{n-2}\cdot\frac{n-1}{n}, n\text{为正奇数}\end{cases}$$

无穷限积分：

$$\int_a^{+\infty}f(x)dx=\lim_{b\to+\infty}\int_a^{\mathrm{b}}f(x)dx=F(+\infty)-F(a);$$

$$\int_{-\infty}^{b}f(x)dx=\lim_{a\to-\infty}\int_a^{\mathrm{b}}f(x)dx=F(-\infty)-F(a);$$

$$\int_{-\infty}^{+\infty}f(x)dx=\lim_{b\to+\infty}\int_a^{\mathrm{b}}f(x)dx+\lim_{a\to-\infty}\int_a^{\mathrm{b}}f(x)dx=F(+\infty)-F(-\infty)$$

瑕积分：

$$\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx = F(b) - \lim_{t\to a^+} F(t);$$

$$\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx = \lim_{t\to b^-} F(t) - F(a);$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$\int_a^{+\infty}\frac{1}{x^p}dx, p>1\text{收敛}, p\le 1\text{发散}；\int_a^1\frac{1}{x^p}dx, 0<p<1\text{收敛}, p\ge 1\text{发散}$$

$$\tau(n) = \int_0^{+\infty} e^{-x}x^{n-1}dx = (n-1)!, \tau(n+1) = n\cdot\tau(n) = n!; \tau(1) = 1;$$

$$\tau(\frac{1}{2}) = \sqrt{\pi} \Rightarrow \int_0^{+\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

## 第六章　定积分应用

1、平面图形的面积：

直角坐标情形：$A = \int_a^b |f(x)|dx$；$A = \int_a^b |f(x) - g(x)|dx$；$A = \int_c^d |\varphi(y) - \psi(y)|dy$

参数方程情形：$A = \int_\alpha^\beta \psi(t)d\varphi(t) = \int_\alpha^\beta \psi(t)\varphi'(t)dt; (\varphi(\alpha) = a; \varphi(\beta) = b)$

极坐标情形：$A = \frac{1}{2}\int_\alpha^\beta \rho^2(\theta)d\theta$

2、空间立体的体积：

由截面面积：$V = \int_a^b A(x)dx$

旋转体：绕 x 轴旋转：
$$V = \int_a^b \pi f^2(x)dx; V = \int_a^b \pi[f^2(x) - g^2(x)dx(x\text{为积分变量})$$
$$V = \int_c^d 2\pi y|\varphi(y)|dy; V = \int_c^d 2\pi y|\varphi(y) - \psi(y)|dy(y\text{为积分变量})$$

绕 y 轴旋转：
$$V = \int_a^b 2\pi x|f(x)|dx = \int_a^b 2\pi x|f(x) - g(x)|dx; (x\text{为积分变量})$$
$$V = \int_c^d \pi[\varphi^2(y) - \psi^2(y)]dy(y\text{为积分变量})$$

3、平面曲线的弧长：

$$s = \int_\alpha^\beta \sqrt{\varphi'^2(t) + \psi'^2(t)}dt = \int_a^b \sqrt{1 + f'^2(x)}dx = \int_\alpha^\beta \sqrt{\rho^2(\theta) + \rho'^2(\theta)}d\theta$$

变力做功：$W = \int_a^b F(x)dx$

抽水做功：克服重力做功=质量$\times g\times$高度,$dW = dM\cdot g\cdot h = \rho\cdot dV\cdot g\cdot h$

液体压力做功：压力=压强×面积，$dF = pdA = \rho\cdot g\cdot h\cdot dA$

# 第七章　向量代数与空间解析几何

两点间距离公式 ：

$|M_1-M_2|=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$ ，

$\vec{a}=(a_x,a_y,a_z)=a_x\vec{i}+a_y\vec{j}+a_z\vec{k}$; $\vec{b}=(b_x,b_y,b_z)=b_x\vec{i}+b_y\vec{j}+b_z\vec{k}$

$\vec{a}\pm\vec{b}=(a_x\pm b_x,a_y\pm b_y,a_z\pm b_z)$; $\lambda\vec{a}=(\lambda a_x,\lambda a_y,\lambda a_z)$

$$\cos\alpha=\frac{a_x}{|\vec{a}|}=\frac{a_x}{\sqrt{a_x^2+a_y^2+a_z^2}},$$

方向余弦：$\cos\beta=\dfrac{a_y}{|\vec{a}|}=\dfrac{a_y}{\sqrt{a_x^2+a_y^2+a_z^2}}$，　　单位向量：$\vec{e}_a=\dfrac{\vec{a}}{|\vec{a}|}=(\cos\alpha,\cos\beta,\cos\gamma)$

$$\cos\gamma=\frac{a_z}{|\vec{a}|}=\frac{a_z}{\sqrt{a_x^2+a_y^2+a_z^2}}$$

数量积：$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos(\widehat{\vec{a},\vec{b}})=a_xb_x+a_yb_y+a_zb_z$

$\vec{a}\cdot\vec{a}=\vec{a}^2=|\vec{a}|^2\Rightarrow\vec{i}\cdot\vec{j}=\vec{j}\cdot\vec{k}=\vec{k}\cdot\vec{i}=0$，$\vec{i}\cdot\vec{i}=\vec{j}\cdot\vec{j}=\vec{k}\cdot\vec{k}=1$

夹角余弦：$\cos(\widehat{\vec{a},\vec{b}})=\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\dfrac{a_xb_x+a_yb_y+a_yb_y}{\sqrt{a_x^2+a_y^2+a_z^2}\sqrt{b_x^2+b_y^2+b_z^2}}$

向量积：$\vec{a}\times\vec{b}=(a_yb_z-a_zb_y)\vec{i}+(a_zb_x-a_xb_z)\vec{j}+(a_xb_y-a_yb_x)\vec{k}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\a_x&a_y&a_z\\b_x&b_y&b_z\end{vmatrix}$

$\vec{a}\times\vec{a}=\vec{0}$，$|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin(\widehat{\vec{a},\vec{b}})=S_{\text{平行四边形}}$，

空间位置关系：$\vec{a}/\!/\vec{b}\Leftrightarrow\vec{a}\times\vec{b}=\vec{0}\Leftrightarrow(\exists\alpha,\beta)\alpha\vec{a}+\beta\vec{b}=\vec{0}\Leftrightarrow\dfrac{b_x}{a_x}=\dfrac{b_y}{a_y}=\dfrac{b_z}{a_z}$

$\vec{a}\perp\vec{b}\Leftrightarrow\vec{a}\cdot\vec{b}=\vec{0}\Leftrightarrow a_xb_x+a_yb_y+a_zb_z=0\Leftrightarrow|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$

平 面 的 方 程 ： 点 法 式 ： $A(x-x_0)+B(y-y_0)+C(z-z_0)$ ； 一 般 式 ：

$Ax+By+Cz+D=0$

截距式：$\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$

两平面的夹角：$\cos\theta=\frac{\left|\overrightarrow{n_1}\cdot\overrightarrow{n_2}\right|}{\left|\overrightarrow{n_1}\right|\left|\overrightarrow{n_2}\right|}=\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}$

点到平面的距离：$d=\frac{\left|Ax_0+By_0+Cz_0+D\right|}{\sqrt{A^2+B^2+C^2}}$

两平行平面的距离：$d=\frac{\left|D_1-D_2\right|}{\sqrt{A^2+B^2+C^2}}$

直线与平面的夹角：$\sin\varphi=\frac{\left|\vec{n}\cdot\vec{s}\right|}{\left|\vec{n}\right|\left|\vec{s}\right|}=\frac{Am+Bn_2+Cp}{\sqrt{A^2+B^2+C^2}\sqrt{m^2+n^2+p^2}}$

空间曲线$C$，曲线的投影$C_{xoy}$，空间立体$\Omega$，曲面$\Sigma$，曲面的投影$D_{xy}$

球面：$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=R^2$

椭圆柱面：$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$；双曲柱面：$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$；抛物柱面：$x^2=2py$

旋转曲面：圆柱面：$x^2+y^2=a^2$；圆锥面：$z^2=b^2(x^2+y^2)$；双叶双曲面：

$\frac{x^2}{a^2}-\frac{y^2+z^2}{c^2}=1$

单叶双曲面：$\frac{x^2+y^2}{a^2}-\frac{z^2}{c^2}=1$；旋转椭球面: $\frac{x^2+y^2}{a^2}+\frac{z^2}{c^2}=1$；旋转抛物面：

$x^2+y^2=2pz$

二次曲面：

椭球面：$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\quad(a>0,b>0,c>0)$

抛物面：椭圆抛物面：$\frac{x^2}{a^2}+\frac{y^2}{b^2}=z$；双曲抛物面：$\frac{x^2}{a^2}-\frac{y^2}{b^2}=z$

单叶双曲面：$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$；双叶双曲面：$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$

椭圆锥面：$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$

总结

## 求极限方法：

1、 极限定义；2、函数的连续性；3、极限存在的充要条件；4、两个准则；
5、两个重要极限；6、等价无穷小；7、导数定义；8 利用微分中值定理；
9、洛必达法则；10、麦克劳林公式展开；

## 求导法：

1、导数的定义（求极限）；2、导数存在的充要条件；3、基本求导公式；
4、导数四则运算及反函数求导；5、复合函数求导；6、参数方程确定的函数求导；
7、隐函数求导法；8、高阶导数求导法（莱布尼茨公式/常用的高阶导数）；

## 等式与不等式的证明：

1、利用微粉中值定理；2、利用泰勒公式展开；3、函数的单调性；
4、最大最小值；5、曲线的凸凹性

# 第八章　多元函数微分法及其应用

一、定义：

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}=\left.\frac{d}{dx}f(x,y_0)\right|_{x=x_0}=f_x(x_0,y_0)=f_x(x,y)\big|_{(x_0,y_0)}$$

二、 微分：

$$\lim_{\rho\to 0}\frac{\Delta z-f_x(x,y)\Delta x-f_y(x,y)\Delta y}{\rho}=0\Leftrightarrow \text{可微，偏导连续}\Rightarrow\text{可微}\Rightarrow\text{连续+偏导存在，}$$

全微分：$dz=f_x(x,y)dx+f_y(x,y)dy$

三、隐函数求导：

$1^{\circ}$　$F(x,y)=0\Rightarrow y=f(x)$且$\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{F_x}{F_y}.$

$2^{\circ}$　$F(x,y,z)=0\Rightarrow z=f(x,y)$且

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z},\ \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$$

四、曲线的切线和法平面

1、曲线方程 $L:\begin{cases}x=\varphi(t)\\ y=\psi(t)\\ z=\omega(t)\end{cases}$，切线：$\dfrac{(x-x_0)}{\varphi'(t_0)}=\dfrac{(y-y_0)}{\psi'(t_0)}=\dfrac{(z-z_0)}{\omega'(t_0)}$，法平面：

$\varphi'(t_0)(x-x_0)+\psi'(t_0)(y-y_0)+\omega'(t_0)(z-z_0)=$

2、曲线方程 $L:\begin{cases} y = y(x) \\ z = z(x) \end{cases}$，切线：$\dfrac{x-x_0}{1} = \dfrac{y-y_0}{y'(x_0)} = \dfrac{z-z_0}{z'(x_0)}$，法平面：

$(x-x_0)+y'(x_0)(y-y_0)+z'(x_0)(z-z_0)$

3、曲线方程 $L:\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$，切向量 $\vec{T} = \pm\left\{F_x, F_y, F_z\right\}_{M_0} \times \left\{G_x, G_y, G_z\right\}_{M_0}$，切线：

$$\frac{x-x_0}{\begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}_{M_0}} = \frac{y-y_0}{\begin{vmatrix} F_z & F_x \\ G_z & G_x \end{vmatrix}_{M_0}} = \frac{z-z_0}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}_{M_0}}$$

四、曲面的切平面和法线

1、曲面方程：$F(x,y,z)=0$，法向量：$\vec{n} = \pm\left\{F_x, F_y, F_z\right\}_{M_0}$，切平面：$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0$，法线：

$$\frac{(x-x_0)}{F_x(x_0,y_0,z_0)} = \frac{(y-y_0)}{F(x_0,y_0,z_0)} = \frac{(z-z_0)}{F(x_0,}$$

2、曲面方程：$z=f(x,y)$，切平面

$f_x(x_0\ y\ z\ \ x-x_0\ + f_y\ x_0\ y\ z\ \ y-y_0 - (z-z = ,$

法线：$\dfrac{x-x_0}{f_x(x_0,y_0)} = \dfrac{y-y_0}{f_y(x_0,y_0)} = \dfrac{z-z_0}{-1}$

五、方向导数：$\left.\dfrac{\partial f}{\partial l}\right|_{M_0} = f_x\big|_{M_0}\cos\alpha + f_y\big|_{M_0}\cos\beta + f_z\big|_{M_0}\cos\gamma$

梯度：$\mathrm{grad}u\big|_{M_0} = \left\{f_x, f_y, f_z\right\}_{M_0}$

第九章：重积分

一、 二重积分：

$$\iint_D f(x,y)d\sigma = \iint_D f(x,y)dxdy = \int_a^b dx\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy = \int_c^d dy\int_{\psi_1(x)}^{\psi_2(x)} f(x,y)dx$$

$$\iint_D f(\rho\cos\theta,\rho\sin\theta)\rho d\rho d\theta = \int_\alpha^\beta d\theta\int_{\varphi_1(\theta)}^{\varphi_2(\theta)} f(\rho\cos\theta,\rho\sin\theta)\rho d\rho$$

二、三重积分：

1、直角坐标系： $\iiint\limits_{\Omega} f(x,y,z)\mathrm{d}V = \iint\limits_{D_{xy}} \mathrm{d}x\mathrm{d}y\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\mathrm{d}z$

$$\iiint\limits_{\Omega} f(x,y,z)dv = \int_{c_1}^{c_2} dz \iint\limits_{D(z)} f(x,y,z)dxdy.$$

2、柱面坐标系： $\begin{cases} x = r\cos\theta, \\ y = r\sin\theta,\ dv = rdrd\theta dz, \\ z = z. \end{cases}$

$$\iiint\limits_{\Omega} f(x,y,z)dv = \int_{\alpha}^{\beta} \mathrm{d}\theta \int_{\rho_1(\theta)}^{\rho_2(\theta)} \mathrm{d}r \int_{z_1(\rho,\theta)}^{z_2(\rho,\theta)} f(\rho\cos\theta,\rho\sin\theta,z)\rho \mathrm{d}z\ .$$

3、球面坐标系：

$$\begin{cases} x = r\sin\varphi\cos\theta, \\ y = r\sin\varphi\sin\theta,\ dv = r^2\sin\varphi drd\varphi d\theta, \\ z = r\cos\varphi. \end{cases}$$

$$\iiint\limits_{\Omega} f(x,y,z)dxdydz = \int_{\alpha}^{\beta} \mathrm{d}\theta \int_{\varphi_1(\theta)}^{\varphi_2(\theta)} \mathrm{d}\varphi \int_{r_1(\theta,\varphi)}^{r_2(\theta,\varphi)} f(r\sin\varphi\cos\theta, r\sin\varphi\sin\theta, r\cos\varphi) r^2 \sin\varphi \mathrm{d}r.$$

二、重积分的应用：

1、体积： $V = \iiint\limits_{\Omega} \mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint\limits_{D_{xy}} [z_2(x,y) - z_1(x,y)]\mathrm{d}x\mathrm{d}y$

2、曲面Σ： $z = f(x,y)$ 面积： $S = \iint\limits_{D_{xy}} \sqrt{1 + f_x'^2(x,y) + f_y'^2(x,y)}\,\mathrm{d}x\mathrm{d}y$

3、质量： $M = \iint\limits_{D} \rho(x,y)\mathrm{d}\sigma$ 或 $M = \iiint\limits_{\Omega} \mu(x,y,z)\ dv$

4、质心 $(\overline{x},\overline{y})$：

$$\overline{x} = \frac{\iint\limits_{D} x\rho(x,y)d\sigma}{M}, \overline{y} = \frac{\iint\limits_{D} y\rho(x,y)d\sigma}{M} \text{或}$$

$$\overline{x} = \frac{\iiint\limits_{\Omega} x\mu(x,y,z)\ dv}{\iiint\limits_{\Omega} \mu(x,y,z)\ dv}, \overline{y} = \frac{\iiint\limits_{\Omega} y\mu(x,y,z)\ dv}{\iiint\limits_{\Omega} \mu(x,y,z)\ dv}, \overline{z} = \frac{\iiint\limits_{\Omega} z\mu(x,y,z)\ dv}{\iiint\limits_{\Omega} \mu(x,y,z)\ dv}$$

5、 转动惯量： $I_x = \iint\limits_{D} y^2\rho(x,y)d\sigma, I_y = \iint\limits_{D} x^2\rho(x,y)d\sigma, I_o = \iint\limits_{D} (x^2+y^2)\rho(x,y)d\sigma$

$$I_x=\iiint\limits_{\Omega}(y^2+z^2)\mu(x,y,z)\ dv, I_y=\iiint\limits_{\Omega}(z^2+x^2)\mu(x,y,z)\ dv$$
或
$$I_z=\iiint\limits_{\Omega}(x^2+y^2)\mu(x,y,z)\ dv, I_o=\iiint\limits_{\Omega}(x^2+y^2+z^2)\mu(x,y,z)\ dv$$

第十章：曲线积分和曲面积分

一、第一类曲线积分：（对弧长的曲线积分）：

$$\int\limits_L f(x,y)ds=\int_\alpha^\beta f(\varphi(t),\psi(t))\sqrt{\varphi'^2(t)+\psi'^2(t)}dt=\int_a^b f(x,y(t))\sqrt{1+y^2(t)}dx$$

$$=\int_\alpha^\beta f(\rho(\theta)\cos\theta,\rho(\theta)\sin\theta)\sqrt{\rho^2(\theta)+\rho'^2(\theta)}d\theta$$

$$\int\limits_L f(x,y,z)ds=\int_\alpha^\beta f(\varphi(t),\psi(t),\omega(t))\sqrt{\varphi'^2(t)+\psi'^2(t)+\omega'^2(t)}dt$$

二、第二类曲线积分（对坐标的曲线积分）：

1、计算公式：

$$\int\limits_L P(x,y)dx+Q(x,y)dy=\int\limits_L[P(x,y)\cos\alpha+Q(x,y)\cos\beta]ds$$

$$=\int_a^b[P(\varphi(t),\psi(t))\varphi'(t)+Q(\varphi(t),\psi(t))\psi'(t)]dt$$

2、格林公式：

$$\iint\limits_D(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial x})dxdy=\int\limits_{\partial D^+}Pdx+Qdy=\int\limits_{\partial D}(P\cos\alpha+Q\cos\beta)ds$$

3、Stokes 公式：

Stokes 公式：$\int\limits_{\Gamma=\partial\Sigma^+}P\mathrm{d}x+Q\mathrm{d}y+R\mathrm{d}z=$

$$\iint\limits_\Sigma\begin{vmatrix}\mathrm{d}y\mathrm{d}z & \mathrm{d}z\mathrm{d}x & \mathrm{d}x\mathrm{d}y\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}=\iint\limits_\Sigma\begin{vmatrix}\cos\alpha & \cos\beta & \cos\gamma\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}dS=\pm\iint\limits_{D_{xy}}f(x,y,z)dxdy$$

4、封闭曲线围城的面积：$A=\frac{1}{2}\int\limits_{\partial D^+}xdy-ydx$

三、第一类曲面积分：

$\Sigma:z=z(x,y)$：$\iint\limits_\Sigma f(x,y,z)dS=\iint\limits_{D_{xy}}f(x,y,z(x,y))\sqrt{1+Z_x^2+Z_y^2}dxdy$

四、第二类曲面积分：

1、计算公式：

$$\iint_{\Sigma} \vec{F}(x,y,z)d\vec{S} = \iint_{\Sigma} P(x,y,z)dydz + Q(x,y,z)dzdx + R(x,y,z)dxdy$$

$$= \iint_{\Sigma} \vec{F}(x,y,z)\cdot \vec{e}_n dS = \iint_{\Sigma} (P\cos\alpha + Q\cos\beta + R\cos\gamma)dS$$

$$\iint_{\Sigma上侧} R(x,y,z)\mathrm{d}x\mathrm{d}y = \iint_{D_{xy}} R[x,y,z(x,y)]\mathrm{d}x\mathrm{d}y; \iint_{\Sigma下侧} R(x,y,z)\mathrm{d}x\mathrm{d}y = -\iint_{D_{xy}} R[x,y,z(x,y)]\mathrm{d}x\mathrm{d}y$$

$$\iint_{\Sigma} P(x,y,z)dydz = \pm\iint_{Dyz} p(x(y,z),y,z)dydz; \quad \iint_{\Sigma} Q(x,y,z)dzdx = \pm\iint_{Dzx} p(x,y(z,x),z)dzdx$$

2、投影转化法：

$$\Sigma: z = z(x,y), dydz = \frac{\cos\alpha}{\cos\gamma}dxdy = -z_x dxdy, dzdx = \frac{\cos\beta}{\cos\gamma}dxdy = -z_y dxdy$$

$$\Sigma: F(x,y,z) = 0, dydz = \frac{F_x}{F_z}dxdy, dzdx = \frac{F_y}{F_z}dxdy$$

3、高斯公式：

$$\iint_{\Sigma} P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y = \iint_{\Sigma} (P\cos\alpha + Q\cos\beta + R\cos\gamma)\mathrm{d}S$$

$$= \pm\iiint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\mathrm{d}V。\text{（}\Sigma\text{为}\partial\Omega^{+}\text{外侧时取}+;\Sigma\text{为}\partial\Omega^{-}\text{内侧时取}-.\text{)}$$

4 $\vec{A}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$，$u = u(x,y,z) \Rightarrow$

散度：$div\vec{A} = P_x + Q_y + R_z$;梯度：$gradu = (u_x, u_y, u_z)$

$div(gradu) = u_{xx} + u_{yy} + u_{zz}$;旋度：$\mathrm{rot}\vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$

## 第十一章 无穷级数

### 一、常数项级数 $\sum_{n=1}^{\infty} u_n$

1、常用级数：等比级数/几何级数:$\sum_{n=0}^{\infty} q^n \begin{cases} 收 = \dfrac{1}{1-q} & |q| < 1 \\ 发 & |q| \geq 1 \end{cases}$

$P$级数:$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} 收 & P > 1 \\ 发 & 0 < P \leq 1 \end{cases}$; 交错$P$级数:$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^p}$收敛$\begin{cases} 绝对收敛 & P > 1 \\ 条件收敛 & 0 < P \leq 1 \end{cases}$

2、正项级数：$u_n \geq 0$

基本定理：收敛$\Leftrightarrow$部分和有上届$S_n < \sigma$

比较审敛法：大收小收，小发大发

比较审敛法的极限形式：　同阶：同收同发；低阶：同收；高阶：同发

比值/根值审敛法：$\rho = \lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n}\,(\rho = \lim\limits_{n\to\infty}\sqrt{u_n}) \Rightarrow \begin{cases} <1, & \text{收敛} \\ >1, & \text{发散} \\ =1, & \text{失效} \end{cases}$

3、交错级数：$\sum\limits_{n=1}^{\infty}(-1)^{n-1}u_n\,(u_n \geq 0)$

莱布尼茨审敛法：$\begin{cases} u_{n+1} \leq u_n \\ \lim\limits_{n\to\infty} u_n = 0 \end{cases} \Rightarrow$ 级数收敛，$S \leq u_1, |r_n| \leq u_{n+1}$

绝对收敛：$\sum\limits_{n=1}^{\infty}|u_n|$收敛$\Rightarrow \sum\limits_{n=1}^{\infty}u_n$收敛，条件收敛：$\sum\limits_{n=1}^{\infty}u_n$收敛而$\sum\limits_{n=1}^{\infty}|u_n|$发散，发散

4、任意项级数：

- 利用定义：部分和有极限$\lim\limits_{n\to\infty}S_n = \begin{cases} S, & \text{收敛} \\ \infty, & \text{发散} \end{cases}$；
- 利用收敛的必要条件：$\lim\limits_{n\to\infty}u_n \neq 0 \Rightarrow$ 发散；
- 利用正项级数（比值/根植）审敛法：

$$\rho = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|(\rho = \lim_{n\to\infty}\sqrt[n]{|u_n|}) \Rightarrow \begin{cases} <1, & \text{绝对收敛} \Rightarrow \text{收敛} \\ >1, & \text{绝对值发散} \Rightarrow \text{发散} \\ =1, & \text{失效} \end{cases}$$

二、幂级数：$\sum\limits_{n=0}^{\infty}a_n(x-x_0)^n$

1、收敛半径：$\rho = \lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|(\rho = \lim\limits_{n\to\infty}\sqrt[n]{|u_n|}) \Rightarrow R = \begin{cases} 1/\rho, & 0<\rho<\infty \\ 0, & \rho = \infty \\ \infty, & \rho = 0 \end{cases}$

2、常用等式：

$$\sum_{n=0}^{\infty}x^n = \frac{1}{1-x}\,(|x|<1),\ \sum_{n=1}^{\infty}x^n = \frac{x}{1-x}\,(|x|<1),\ \sum_{n=0}^{\infty}(-1)^n x^n = \frac{1}{1+x}\,(|x|<1)$$

$$\sum_{n=1}^{\infty}\frac{x^n}{n} = -\ln(1-x)\quad(-1 \leq x < 1),\ \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = \ln(1+x)\quad(-1 < x \leq 1)$$

$$\sum_{n=0}^{\infty}(n+1)x^n = \sum_{n=1}^{\infty}nx^{n-1} = \frac{1}{(1-x)^2}\quad(|x|<1)$$

，

$$\sum_{n=0}^{\infty}\frac{1}{2n+1}x^{2n+1}=\sum_{n=1}^{\infty}\frac{1}{2n-1}x^{2n-1}=\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| \quad (|x|<1)$$

$$\arctan x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}(|x|<1)$$

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots;\quad x\in(-\infty,+\infty)$$

$$\sin x=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots+\frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}+\cdots;\quad x\in(-\infty,+\infty)$$

$$\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots;\quad x\in(-\infty,+\infty)$$

$$\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+(-1)^{n-1}\frac{x^n}{n}+\cdots;\quad x\in(-1,\ 1]$$

$$(1+x)^{\alpha}=1+\sum_{n=1}^{\infty}\frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}x^n$$

$$=1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\cdots+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n+\cdots;\quad x\in(-1,\ 1)$$

3、泰勒展开：

$$f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n,a_n=\frac{1}{n!}f^{(n)}(x_0),R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1},(\xi\in(x_0,x))$$

$$\Leftrightarrow\lim_{n\to\infty}R_n(x)=0$$

三、傅里叶级数：$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$

1、$T=2\pi$：$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=S(x)$,

（$x\in(-\infty,+\infty)$，且$x\neq$间断点）

其中$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx,(n=0,1,2\cdots)$； $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx,(n=1,2\cdots)$。

(间断点处，$S(x)=\frac{f(x^-)+f(x^+)}{2}$)

若$f(x)$为奇函数$\Rightarrow$正弦级数（$a_n=0,b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx\mathrm{d}x$)；

若$f(x)$为偶函数$\Rightarrow$余弦级数（$b_n=0,a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\mathrm{d}x$)；

2、$T=2l$：$f(x)=\dfrac{a_0}{2}+\sum\limits_{n=1}^{\infty}(a_n\cos\dfrac{n\pi x}{l}+b_n\sin\dfrac{n\pi x}{l})$，　（$x\in(-\infty,+\infty)$，且$x\neq$间断点）

其中$a_n=\dfrac{1}{l}\int_{-l}^{l}f(x)\cos\dfrac{n\pi x}{l}dx,(n=0,1,2\cdots)$；$b_n=\dfrac{1}{l}\int_{-l}^{l}f(x)\sin\dfrac{n\pi x}{l}dx,(n=1,2\cdots)$。

3、非周期函数$f(x)$,

(1)$x\in[-l,l]$：$f(x)\xrightarrow{\text{周期延拓}}F(x)$展开$\rightarrow$限制

$$f(x)=S(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}),(x\in(-l,l))$$

($x=\pm l$时，$S(x)=\dfrac{f(-l^+)+f(l^-)}{2}$)

(2)$x\in[0,l]$：$f(x)\xrightarrow{\text{奇延拓/偶延拓}}\xrightarrow{\text{周期延拓}}F(x)$展开$\rightarrow$限制

奇延拓：$f(x)=\sum\limits_{n=1}^{\infty}b_n\sin\dfrac{n\pi x}{l},(x\in(0,l))$;

$b_n=\dfrac{2}{l}\int_0^l f(x)\sin\dfrac{n\pi x}{l}dx$　($n=1,\ 2,\cdots$);($x=0$或$l$时，$S(x)=0$);

偶延拓：$f(x)=\dfrac{a_0}{2}+\sum\limits_{n=1}^{\infty}a_n\cos\dfrac{n\pi x}{l}(x\in[0,l]$ )

$a_n=\dfrac{2}{l}\int_0^l f(x)\cos\dfrac{n\pi x}{l}dx$　($n=0,\ 1,\ 2,\cdots$)，端点处不间断。

## 第十二章　微分方程

### 一、基本类型的一阶微分方程:

1、可分离变量方程：$\dfrac{dy}{dx}=f(x)g(y)$，分离变量，两边积分$\int\dfrac{dy}{g(y)}=\int f(x)\,dx$

2、一阶线性微分方程：$\dfrac{dy}{dx}+P(x)y=Q(x)$

$$\begin{cases}Q(x)=0 & \text{齐次:通解：}y=e^{-\int P(x)dx},\\ Q(x)\neq0 & \text{非齐次:通解：}y=e^{-\int P(x)dx}(\int Q(x)e^{\int P(x)dx}dx+C)\end{cases}$$

3、全微分方程：$P(x,y)\mathrm{d}x+Q(x,y)\mathrm{d}y=0$(其中$P_y=Q_x$)

通解：$u(x,y)=C$. (1)、分项组合法;

(2)、特殊路径法：$u(x,y)=\int_{x_0}^{x}P(x,y_0)\mathrm{d}x+\int_{y_0}^{y}Q(x,y)\mathrm{d}y=C$.

(3)、偏积分法;

$$\left.\begin{array}{l}\mathrm{P(x,\ y)}=\dfrac{\partial \mathrm{u}}{\partial \mathrm{x}}\Rightarrow \mathrm{u(x,y)}=\int \mathrm{P(x,y)dx}+\mathrm{c(y)}\\ \mathrm{Q(x,\ y)}=\dfrac{\partial \mathrm{u}}{\partial \mathrm{x}}\Rightarrow \mathrm{c'(y)}=\mathrm{Q}-\dfrac{\partial}{\partial \mathrm{y}}\int \mathrm{P(x,y)dx}=\varphi(\mathrm{y})\end{array}\right\}\Rightarrow \mathrm{u(x,y)}=\int \mathrm{P(x,y)dx}+\int\varphi(\mathrm{y})\mathrm{dy}$$

二、可化为基本类型的一阶微分方程:

⑴ 齐次方程:$\frac{dy}{dx}=f(\frac{y}{x})$或$\frac{dy}{dx}=f(\frac{a_1x+b_1y}{a_2x+b_2y})$,令$u=\frac{y}{x}$

(2) 准齐次方程:$\frac{dy}{dx}=f(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2})$

$$\begin{cases} 若\Delta=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\neq 0, 令\begin{cases} x=X+h \\ y=Y+k \end{cases}，(h,k由\begin{cases} a_1x+b_1y+c_1=0 \\ a_2x+b_2y+c_2 \end{cases}解得) \\ \qquad\qquad \Rightarrow \frac{dY}{dX}=f(\frac{a_1X+b_1Y}{a_2X+b_2Y})，再令u=\frac{Y}{X}。 \\ 若\Delta=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=0, \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{k(a_1x+b_1y)+c_1}{a_1x+b_1y+c_2}=f(a_1x+b_1y).令u=a_1x+b_1y。 \end{cases}$$

(3)$\frac{dy}{dx}=f(ax+by+c)$　　令 $u=ax+by+c$。

(4)伯努利方程：$\frac{dy}{dx}+P(x)y=Q(x)y^{\alpha}$ $(\alpha\neq 0,1)$，令 $z=y^{1-\alpha}\Rightarrow\frac{dz}{dy}+(1-\alpha)P(x)z=(1-\alpha)Q(x)$

(5) $P(x,y)dx+Q(x,y)dy=0$　(其中$P_y\neq Q_x$)$\Rightarrow\frac{dy}{dx}=-\frac{P(x,y)}{Q(x,y)}$

(6) 关于$x$的线性方程/伯努利方程:

$\frac{dx}{dy}+P(y)x=Q(y)$;　　$\frac{dx}{dy}+P(y)x=Q(y)x^{\alpha}$,令$z=x^{1-\alpha}$

(7) $P(x,y)dx+Q(x,y)dy=0$ (其中$P_y\neq Q_x$)

求积分因子方法:

1、分项组合法：常用全微分公式;

2、公式法:

(1)方程有形如$u(x)$的积分因子$\Leftrightarrow\frac{1}{Q}(P_y-Q_x)=\varphi(x)\Rightarrow u(x)=ce^{\int\varphi(x)dx}$

(2)方程有形如$u(y)$的积分因子$\Leftrightarrow\frac{1}{P}(P_y-Q_x)=\psi(y)\Rightarrow u(y)=ce^{-\int\psi(y)dy}$

(3)齐次方程的积分因子$u(x,y)=\frac{1}{xP+yQ}$

三、可降阶的高阶微分方程:

(1) $\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = f(x)$ 连续积分n次；

（2）$y'' = f(x, y')$, 令 $y' = p$，则 $y'' = p' \Rightarrow p' = f(x, p)$

（3）$y'' = f(y, y')$ ,令 $y' = p$，则 $y'' = p\frac{\mathrm{d}p}{\mathrm{d}y} \Rightarrow p\frac{\mathrm{d}p}{\mathrm{d}y} = f(y, p)$

## 四、二阶常系数齐次线性微分方程

$y'' + py' + qy = 0 \Leftrightarrow$ 特征方程：$r^2 + pr + q = 0$

$\Delta = p^2 - 4q > 0, r_1 \neq r_2 \Rightarrow$ 通解：$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$

$\Delta = p^2 - 4q = 0, r_1 = r_2 \Rightarrow$ 通解：$y = (C_1 + C_2 x)e^{r_1 x}$

$\Delta = p^2 - 4q < 0, r_{1,2} = \alpha \pm i\beta \Rightarrow$ 通解：$y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$

## 四、二阶常系数非齐次线性微分方程

$y'' + p\,y' + q\,y = f(x)$ 通解$y(x) =$ 齐次通解$Y(x) +$ 非齐次特解$y^*(x)$

(1) $f(x) = e^{\lambda x} P_m(x) \Rightarrow$ 特解形式$y^* = x^k Q_m(x) e^{\lambda x} \begin{pmatrix} \lambda\text{不是特征根} & k = 0 \\ \lambda\text{是特征单根} & k = 1 \\ \lambda\text{是特征重根} & k = 2 \end{pmatrix}$

(2) $f(x) = f(x) = e^{\lambda x}\left[P_l(x)\cos\omega x + P_n(x)\sin\omega x\right]$

$\Rightarrow$ 特解形式$y^* = xe^{\lambda x}\left[R_m^{(1)}(x)\cos\omega x + R_m^{(2)}(x)\sin\omega x\right] \begin{pmatrix} \lambda + iw\text{不是特征根} & k = 0 \\ \lambda + iw\text{是特征根} & k = 1 \end{pmatrix}$